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*Mechanicam vero duplicem Veteres constituerunt: Rationalem quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique Mechanica nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut Mechanica omnis a Geometria ita distinguatur, ut quicquid accuratum sit ad Geometriam referatur, quicquid minus accuratum ad Mechanicam. Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfector est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.*

NEWTON

*La généralité que j'embrasse, au lieu d'éblouir nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité.*

EULER

*Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ...*

LAGRANGE

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# *The Shape of the Strongest Column*

JOSEPH B. KELLER

*Communicated by C. TRUESDELL*

## **Abstract**

The problem of determining that shape of column which has the largest critical buckling load is solved, assuming that the length and volume are given and that each cross section is convex. The strongest column has an equilateral triangle as cross section, and it is tapered along its length, being thickest in the middle and thinnest at its ends. Its buckling load is 61.2% larger than that of a circular cylinder. For columns all of whose cross sections are similar and of prescribed shape—not necessarily convex—the best tapering is found to increase the buckling load by one third over that of a uniform column. This result, which was independently obtained by H. F. WEINBERGER, is originally due to CLAUSEN (1851). For a uniform column, triangularizing is shown to increase the buckling load by 20.9% over that of a circular cylinder. The results lead to isoperimetric inequalities for the buckling loads of arbitrary columns.

## **1. Introduction**

Of all columns having prescribed length and volume and convex cross sections, which is the strongest, *i.e.*, which has the largest critical buckling load? We shall solve this problem and show that the strongest column is not uniform but is tapered, being thickest at its center and thinnest at its ends. We shall also show that the cross section of the strongest column is not a circle but is instead an equilateral triangle. Appropriately tapering a column of circular cross section increases its critical buckling load by one third over that of a uniform column. Changing the cross section from a circle to an equilateral triangle increases the critical buckling load by 20.9%. Thus the combined modifications of tapering and triangularizing increase the critical buckling load by 61.2% over that of a circular cylinder.

Our principal result can also be stated as an isoperimetric inequality. In this form it states that the critical buckling load of any column is less than a given function of the length, volume and YOUNG'S modulus of the column. The given function is just the critical buckling load of the strongest column with the same length, volume and YOUNG'S modulus. In the course of our analysis two other isoperimetric inequalities arise. One pertains to an untwisted tapered column with similar cross sections and the other to cylindrical columns. The former can also be stated as an isoperimetric inequality for a certain eigenvalue

problem, without reference to columns. The latter is related to a geometrical problem which is not completely solved. In the last section we present some results for the nonlinear buckling problem.

For the sake of clarity we first determine the strongest cylindrical column of arbitrary convex cross section. Then we determine the strongest untwisted column, all cross sections of which are similar. Upon combining these two results we find the strongest untwisted column with convex cross sections. Finally we show that this column is also strongest among all columns with convex cross sections. This procedure is also economical since we use the simple results in the final section.

Our analysis is based upon the six equations of equilibrium of a column of arbitrary shape. To these we add expressions for the three stress couples in terms of the curvatures and twist of the column. These three relations are the usual generalizations of the Bernoulli-Euler theory. We then seek solutions of the nine equations subject to the conditions that the stress couples vanish at the ends of the column and that the applied load have a prescribed value. To determine the critical buckling load we linearize the equations about the straight unbuckled state. The linearized problem is an eigenvalue problem and has non-zero solutions only for special values of the load. The smallest of these is the critical buckling load. It depends upon the shape of the column. We next apply variational techniques to determine that shape which maximizes the critical buckling load. In this way we obtain the results described above.

I wish to thank CLIFFORD TRUESDELL for proposing this problem to me. He also suggested it to HANS F. WEINBERGER, who independently obtained the results of Section 3 on untwisted columns with circular cross sections. However, as TRUESDELL later pointed out, that result had been obtained by CLAUSEN<sup>1</sup> in 1851. LAGRANGE<sup>2</sup> in 1773 first treated that problem but arrived at the wrong result due to computational errors<sup>3</sup>.

## 2. Cylindrical columns of arbitrary convex cross section

For simplicity let us begin by considering cylindrical columns of arbitrary cross section. Let  $L$  and  $V$  denote the length and volume of such a column and  $A = V/L$  its cross sectional area. Let the  $x$ -axis pass through the centroid of each cross section of the column in its straight or unbuckled state. Suppose that  $I'$  is the largest moment of inertia of the cross section about a line in its

<sup>1</sup> Über die Form architektonischer Säulen. Bulletin physico-mathématique de l'Académie 9, 368–379 (St. Pétersbourg 1851). Also Mélanges Mathématiques et Astronomiques, Tome I (1849–1853), pp. 279–294. St. Petersburg 1853. This work is summarized in TODHUNTER, I., & K. PEARSON, A History of the Theory of Elasticity and of the Strength of Materials, vol. 2, pp. 325–329. Cambridge 1893.

<sup>2</sup> Sur la figure des colonnes. Miscellanea Taurinensia (Royal Society of Turin) Tomus V, p. 123, 1770–1773. Also Oeuvres, vol. 2, pp. 125–170. Gauthier-Villars 1868. Summarized in TODHUNTER & PEARSON, vol. 1, pp. 66–67. Cambridge 1886. A more complete early history of this and related problems is given by TRUESDELL, The Rational Mechanics of Flexible or Elastic Bodies, 1638–1788, L. Euleri Opera Omnia II 11<sub>2</sub> (in press).

<sup>3</sup> These errors are pointed out in his Oeuvres by the editor. They are not mentioned in TODHUNTER & PEARSON.

plane through the centroid and that the line is the  $y$ -axis. Similarly let the minimum moment of inertia be  $I$ . Then the corresponding line is the  $z$ -axis.

Let us consider a state of equilibrium of the column in which a compressive load  $T_0$  is applied longitudinally at its ends. Let us further suppose that the deflection of the column from the straight unbuckled state is small. Then the equilibrium is governed by linear equations and every state is a superposition of a deflection in the  $xy$ -plane and another in the  $xz$ -plane. These deflections  $y(x)$  and  $z(x)$  satisfy the equations

$$y_{xx} + \frac{T_0}{EI} y = 0, \quad y(0) = y(L) = 0, \quad (1)$$

$$z_{xx} + \frac{T_0}{EI} z = 0, \quad z(0) = z(L) = 0. \quad (2)$$

The boundary conditions in (1) and (2) apply if the column is pinned at its ends, and  $E$  denotes the YOUNG'S modulus of the column material.

The least positive value of  $T_0$  for which (1) has a non-trivial solution is  $\pi^2 E I L^{-2}$  while for (2) it is  $\pi^2 E I' L^{-2}$ . The critical buckling load is the minimum of these two values. Since by assumption  $I \leq I'$ , we have for the critical load

$$T_0 = \pi^2 E I L^{-2}. \quad (3)$$

The strongest column is that for which  $T_0$  is as large as possible. Since  $E$ ,  $L$  and  $V$ , and therefore  $A$ , are prescribed, the strongest column is one for which  $I$  is largest. But  $I$  can be made arbitrarily large if the cross section is nonconvex. Therefore the problem of finding a strongest column makes sense only if the cross section is required to be convex. Then the cross section of the strongest column is the solution of the following geometrical problem. Problem: Of all plane convex domains of area  $A$ , which has the largest value of  $I$ ? Here  $I$  denotes the minimum value of the moment of inertia of the domain about any line through its centroid.

This problem does not seem to have been investigated previously. It is easy to show that the domain which solves the problem has the same moment of inertia about every line through its centroid which lies in the plane. However this domain is not a circle. This follows from the facts that the circle has the smallest polar moment of inertia of all plane figures of given area and that  $I$  is one half the polar moment for any figure having the same moment  $I$  about every line through the centroid. Every regular polygon has the same moment about every line, through the centroid, and of these the equilateral triangle has the largest  $I$  for given area. Therefore it has been conjectured by PETER UNGAR that the equilateral triangle is the solution of the above problem. GEORGE STELL has proved that it is the solution among all figures having an  $n$ -fold symmetry axis with  $n \geq 3$ . The complete solution has not yet been given, but there is little doubt that it is the equilateral triangle. If this is the case, we have the isoperimetric inequality for convex domains

$$I \leq \frac{A^2}{6\sqrt{3}} = A^2(.096225). \quad (4)$$

Equality holds for the equilateral triangle.

By using (4), (3) yields the isoperimetric inequality for the critical buckling load of cylindrical columns with convex cross sections

$$T_0 \leq \frac{\pi^2 E A^2}{6\sqrt{3} L^2} = \frac{E A^2}{L^2} (.949703). \quad (5)$$

Equality holds for the equilateral triangular cross section. The numerical coefficient in (4) is 20.9% larger than the value  $\pi/4 = .785$  for the circular cross section.

### 3. Tapered untwisted columns with similar cross sections

Let us now consider tapered columns. We shall still assume that all cross sections are similar and that the principal axes of inertia are parallel. Thus the column is not twisted. However the area  $A(x)$  of the cross section may vary with  $x$  subject to the condition

$$\int_0^L A(x) dx = V. \quad (1)$$

The moments  $I$  and  $I'$  are related to  $A$  by the equations

$$I(x) = \alpha A^2(x), \quad I'(x) = \alpha' A^2(x). \quad (2)$$

The constants  $\alpha$  and  $\alpha'$ , with  $\alpha \leq \alpha'$ , depend upon the shape of the cross section. Now (1) becomes

$$y_{xx} + \frac{T_0}{E \alpha A^2(x)} y = 0, \quad y(0) = y(L) = 0. \quad (3)$$

We seek that function  $A(x)$  satisfying (1), which maximizes the lowest eigenvalue  $T_0$  of (3). This function will be the area variation of the strongest column.

Before solving this problem we first introduce the new variable  $\xi = x/L$  and the eigenvalue parameter  $\lambda = T_0 L^2 / \alpha E$  and consider  $y$  and  $A$  as functions of  $\xi$ . Then (3) and (1) become

$$y'' + \lambda A^{-2}(\xi) y = 0, \quad y(0) = y(1) = 0, \quad (4)$$

$$\int_0^1 A(\xi) d\xi = V/L. \quad (5)$$

Now we assume that  $A(\xi)$  is the solution. We introduce a family of functions  $A(\xi, \varepsilon)$  depending differentiably on a parameter  $\varepsilon$ , satisfying (5), and containing the solution  $A(\xi) = A(\xi, 0)$ . Then for each  $\varepsilon$ , (4) will have a solution  $y(\xi, \varepsilon)$  with the corresponding lowest eigenvalue  $\lambda(\varepsilon)$ , both of which depend differentiably on  $\varepsilon$ . Next we differentiate (4) and (5) with respect to  $\varepsilon$ , obtaining

$$y''_\varepsilon + \lambda A^{-2} y_\varepsilon = 2\lambda A_\varepsilon A^{-3} y - \lambda_\varepsilon A^{-2} y, \quad y_\varepsilon(0) = y_\varepsilon(1) = 0, \quad (6)$$

$$\int_0^1 A_\varepsilon d\xi = 0. \quad (7)$$

The derivative  $y_\varepsilon$  satisfies an inhomogeneous linear differential problem (6) and  $y$  satisfies the corresponding homogeneous problem (4). Therefore the right side of the differential equation in (6) must be orthogonal to  $y$ . At  $\varepsilon = 0$ ,  $\lambda_\varepsilon = 0$  since by assumption  $\lambda$  is a maximum at  $\varepsilon = 0$ . The orthogonality condition is

then

$$\int_0^1 A_\epsilon (A^{-3} y^2) d\xi = 0. \quad (8)$$

Since (8) must hold for every  $A_\epsilon(\xi)$  satisfying (7), it follows that  $A^{-3} y^2$  is a constant which we denote by  $c^{\frac{2}{3}}$ . Thus for the strongest column we have

$$y^2 = c^{\frac{2}{3}} A^3. \quad (9)$$

To find  $y$  we first eliminate  $A$  from (4) by means of (9) and obtain

$$y'' + \lambda c y^{-\frac{1}{3}} = 0, \quad y(0) = y(1) = 0. \quad (10)$$

The solution of (10) is

$$\left| \xi - \frac{1}{2} \right| = \frac{1}{2} + \frac{1}{\pi} \left( \frac{y}{y_0} \right)^{\frac{1}{3}} \left[ 1 - \left( \frac{y}{y_0} \right)^{\frac{2}{3}} \right]^{\frac{1}{2}} - \frac{1}{\pi} \sin^{-1} \left( \frac{y}{y_0} \right)^{\frac{1}{3}}. \quad (11)$$

Here  $y_0$  is given by

$$y_0^{\frac{3}{2}} = \frac{2}{3\pi} (\lambda c)^{\frac{1}{2}}. \quad (12)$$

Upon replacing  $y$  in (11) by  $A$ , with the aid of (9), we obtain the following equation for the solution  $A(\xi)$ :

$$\left| \xi - \frac{1}{2} \right| = \frac{1}{2} + \frac{1}{\pi} \left[ \frac{A}{A_0} - \left( \frac{A}{A_0} \right)^2 \right]^{\frac{1}{2}} - \frac{1}{\pi} \sin^{-1} \left( \frac{A}{A_0} \right)^{\frac{1}{2}}. \quad (13)$$

The constant  $A_0 = A(\frac{1}{2})$  in (13) can be determined as a function of  $V/L$  by inserting (13) into (5). This yields

$$V/L = 2 \int_0^{\frac{1}{2}} A(\xi) d\xi = 2 \int_0^{A_0} \xi dA = \frac{3A_0}{4}. \quad (14)$$

To find  $\lambda$  let us use (9) to eliminate  $y$  from (4), which becomes

$$A A'' + \frac{1}{2} (A')^2 + \frac{2}{3} \lambda = 0. \quad (15)$$

From (13) we find that  $A'(\frac{1}{2}) = 0$  and  $A''(\frac{1}{2}) = \pi^2 A_0/2$ . Upon inserting these values into (15) and using (14) for  $A_0$  we obtain

$$\lambda = \frac{4\pi^2}{3} \left( \frac{V}{L} \right)^2. \quad (16)$$

Now from the definition of  $\lambda$  we find the critical buckling load of the strongest column. The result may be stated as an isoperimetric inequality

$$T_0 \leq \frac{4\pi^2}{3} \alpha V^2 L^{-4} E. \quad (17)$$

Equality holds only for the strongest column of given volume, the shape of which is given by (13). For it the critical buckling load is  $\frac{4}{3}$  times as large as that of a cylindrical column of the same volume, length and cross section.

Let us now combine the results of this and the preceding section. Then we find that the strongest untwisted column, all of whose cross sections are similar and convex, is tapered according to (13) and has an equilateral triangular cross section. Upon using the value  $\alpha = (6\sqrt{3})^{-1}$  appropriate to this triangle, we obtain from (17) the isoperimetric inequality

$$T_0 \leq \frac{2\pi^2}{9\sqrt{3}} \frac{V^2 E}{L^4} = \frac{V^2 E}{L^4} (1.26627). \quad (18)$$

Equality holds only for the column just described. For it the critical buckling load is 1.612 times as large as for a circular cylindrical column. In the remaining sections we shall show that (18) also holds for twisted columns, without the restriction that the cross sections be similar.

Our result (16) can be expressed as an isoperimetric inequality for the eigenvalue problem (4) without reference to columns. It can be formulated thus: Let  $\lambda$  be the lowest eigenvalue of the problem

$$y'' + \lambda q(x) y = 0, \quad y(0) = y(L) = 0, \quad q(x) > 0. \quad (19)$$

Then

$$\lambda \leq \frac{4\pi^2}{3L^2} \left[ \int_0^L q^{-\frac{1}{2}}(x) dx \right]^2. \quad (20)$$

Equality holds if  $q(x) = A^2(x/L)$ , where  $A$  is given by (13).

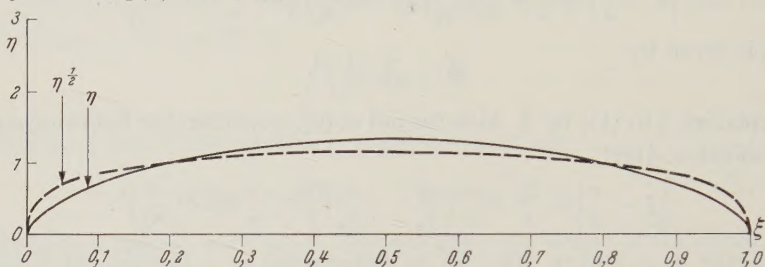


Fig. 1. The area (solid curve) and square root of the area (dashed curve) of the strongest column as functions of distance along the column. These curves are based upon equations (21) and (22). The vertical coordinate is  $\eta = AL/V$  or  $\eta^{\frac{1}{2}} = (AL/V)^{\frac{1}{2}}$ , and the horizontal coordinate is  $\xi = x/L$ . Here  $A(x)$  is the cross sectional area at the distance  $x$  along the column,  $L$  is the length and  $V$  the volume of the column

The equation (13) for the shape of the strongest column can be simply expressed in terms of a parameter  $\vartheta$  which varies from 0 to  $\pi$ . Then, as CLIFFORD TRUESDELL has observed, (13) and (14) yield

$$A = \frac{4V}{3L} \sin^2 \vartheta, \quad (21)$$

$$\xi = \frac{1}{\pi} \left( \vartheta - \frac{1}{2} \sin 2\vartheta \right). \quad (22)$$

#### 4. Arbitrary columns with convex cross sections

Let us now consider any column of length  $L$  and volume  $V$ , all cross sections of which are convex. Among them we seek the strongest one. To find it we begin by considering any such column in a state of equilibrium in the absence of external distributed forces or couples. The vanishing of the net force and torque on each portion of the column leads to six equations [LOVE p. 387 eq. 10, p. 388 eq. 11]<sup>1</sup>

$$N_s - N' \tau + T \kappa' = 0, \quad (1)$$

$$N_s' - T \kappa + N \tau = 0, \quad (2)$$

$$T_s - N \kappa' + N' \kappa = 0, \quad (3)$$

$$G_s - G' \tau + H \kappa' - N' = 0, \quad (4)$$

$$G_s' - H \kappa + G \tau + N = 0, \quad (5)$$

$$H_s - G \kappa' + G' \kappa = 0. \quad (6)$$

<sup>1</sup> LOVE, A. E. H.: A Treatise on the Mathematical Theory of Elasticity, 4<sup>th</sup> ed. Cambridge Univ. Press. 1927.

In these equations  $N$  and  $N'$  are the shearing forces,  $T$  is the tension,  $G$  and  $G'$  are the flexural couples,  $H$  is the torsional couple,  $\kappa$  and  $\kappa'$  are the components of curvature of the central line and  $\tau$  is the twist of the rod. The subscript  $s$  denotes differentiation with respect to arclength along the central line. The forces and couples are expressed as components along the principal torsion-flexure axes of the rod.

In the usual approximate theory of a naturally straight thin rod, which is a generalization of the Bernoulli-Euler theory, it is assumed that the stress couples are related to the curvatures and twist of the rod by the equations [LOVE p. 388, eq. 12, p. 397, eq. 28]<sup>1</sup>

$$G = E I \kappa, \quad (7)$$

$$G' = E I' \kappa', \quad (8)$$

$$H = C (\tau - \tau_0). \quad (9)$$

In these equations  $E$  is the YOUNG'S modulus of the rod material, which is assumed to be isotropic,  $I$  and  $I'$  are the moments of inertia of the cross section of the rod about its principal axes,  $C$  is the torsional rigidity and  $\tau_0$  is the twist in the unstressed state. The nine equations (1)–(9), together with appropriate boundary conditions, suffice to determine the nine quantities  $N, N', T, G, G', H, \kappa, \kappa'$  and  $\tau$ .

We wish to consider the buckling of a naturally straight column subjected to a compressive axial load. In the straight or unbuckled state  $\kappa = \kappa' = 0$ . Then from (7) and (8) it follows that  $G = G' = 0$ . But then (4) and (5) yield  $N = N' = 0$ . Therefore (3) and (4) become  $T_s = H_s = 0$  or  $T = -T_0 = \text{constant}$ ,  $H = \text{constant}$ . We shall assume that no twisting couple is applied at the ends of the rod. Consequently  $H = 0$ , and then (9) shows that  $\tau = \tau_0$ . Thus in the straight state only  $T$  and  $\tau$  differ from zero. If we consider a buckled state which is sufficiently close to the unbuckled state, we may assume that all quantities except  $T$  and  $\tau$  are small while  $T$  and  $\tau$  are close to their values in the unbuckled state. Then we may linearize (1)–(9), which become

$$N_s - N' \tau_0 - T_0 \kappa' = 0, \quad (10)$$

$$N'_s + T_0 \kappa + N \tau_0 = 0, \quad (11)$$

$$T_s = 0, \quad (12)$$

$$G_s - G' \tau_0 - N' = 0, \quad (13)$$

$$G'_s + G \tau_0 + N = 0, \quad (14)$$

$$H_s = 0. \quad (15)$$

From (15) and the boundary condition that no couples act at the ends, it follows that  $H = 0$ , and (9) still shows that  $\tau = \tau_0$ . Similarly (12) yields  $T = \text{constant}$ , and if  $T_0$  is the applied load, then  $T = -T_0$ .

The linearized buckling problem is that of solving (10), (11), (13), (14), (7) and (8) for  $N, N', G, G', \kappa$  and  $\kappa'$  when  $I, I'$  and  $\tau_0$  are given functions of  $s$ , and  $E$  is a given constant. At the endpoints  $s=0$  and  $s=L$  we require that

<sup>1</sup> See the previous footnote.

no couples act. Thus

$$G(0) = G'(0) = G(L) = G'(L) = 0. \quad (16)$$

This is an eigenvalue problem and will have non trivial solutions only for certain special values of the constant  $T_0$ . The smallest buckling load is the least value of  $T_0$  for which a non trivial solution exists.

If we eliminate  $\kappa$  and  $\kappa'$  from (10) and (14) by means of (7) and (8), we obtain

$$N_s - N' \tau_0 - \frac{T_0}{EI'} G' = 0, \quad (17)$$

$$N'_s + N \tau_0 + \frac{T_0}{EI} G = 0. \quad (18)$$

We must now solve (13), (14), (17) and (18) for  $N$ ,  $N'$ ,  $G$  and  $G'$  subject to (16), which is possible only if  $T_0$  is an eigenvalue. We now seek that column, characterized by  $\tau_0$ ,  $I$  and  $I'$ , for which the first eigenvalue is largest. It must have the volume  $V$ , so

$$\int_0^L A(s) ds = V. \quad (19)$$

### 5. The variational problem

Let us now consider a one parameter family of columns depending differentiably upon the parameter  $\varepsilon$  and containing the strongest column at  $\varepsilon = 0$ . Thus  $\tau_0$ ,  $I$  and  $I'$  are all differentiable functions of  $\varepsilon$  as well as of  $s$ . Then the solution of the buckling problem will also depend differentiably on  $\varepsilon$ . Then we obtain from (13), (14), (17), (18), (16) and (19) of Section 4 respectively, with  $\partial/\partial \varepsilon$  denoted by a dot, the variational equations

$$\dot{G}_s - \dot{G}' \tau_0 - \dot{N}' = G' \dot{\tau}_0, \quad (1)$$

$$\dot{G}'_s + \dot{G} \tau_0 + \dot{N} = -G \dot{\tau}_0, \quad (2)$$

$$\dot{N}_s + \dot{N} \tau_0 - \frac{T_0}{EI} \dot{G} = -N \dot{\tau}_0 - \left( \frac{T_0}{EI} \right)' G, \quad (3)$$

$$\dot{N}'_s - \dot{N}' \tau_0 + \frac{T_0}{EI'} \dot{G}' = N' \dot{\tau}_0 + \left( \frac{T_0}{EI'} \right)' G', \quad (4)$$

$$\dot{G}(0) = \dot{G}'(0) = \dot{G}(L) = \dot{G}'(L) = 0, \quad (5)$$

$$\int_0^L \dot{A} ds = 0. \quad (6)$$

The equations (1)–(5) form an inhomogeneous system for the four derivatives  $\dot{G}$ ,  $\dot{G}'$ ,  $\dot{N}$  and  $\dot{N}'$ . The corresponding homogeneous system is just that satisfied by  $G$ ,  $G'$ ,  $N$ , and  $N'$ , *i.e.*, (13), (14), (17), (18) and (16) of Section 4. Therefore in order that the inhomogeneous system have a solution, the right sides of (1) to (4) must be orthogonal to all the solutions of the adjoint homogeneous system. The only solution of this system is  $(-N', N, -G', G)$ . Thus the orthogonality condition is

$$\int_0^L \left[ -2(N' G' + N G) \dot{\tau}_0 - G'^2 \left( \frac{T_0}{EI'} \right)' - G^2 \left( \frac{T_0}{EI} \right)' \right] ds = 0. \quad (7)$$

In (7) we choose  $\dot{I}' = \dot{I} = 0$  and assume that  $\dot{T}_0 = 0$  at  $\varepsilon = 0$ . Then since (7) must hold for every choice of  $\dot{\tau}_0$ , we conclude that for  $\varepsilon = 0$

$$N' G' = -N G. \quad (8)$$

Let us now deduce some conclusions from the foregoing equations. First we multiply (13) of Section 4 by  $-G'$  and (14) of Section 4 by  $G$ , add and use (8) to obtain

$$G'_s G - G' G_s + (G'^2 + G^2) \tau_0 = 0. \quad (9)$$

We rewrite (9) and define  $\beta_s$  as follows:

$$\frac{(G/G')_s}{(G'/G)^2 + 1} = -\tau_0 \equiv -\beta_s. \quad (10)$$

Upon integrating (10) and adding an appropriate constant to  $\beta$ , we have

$$G'/G = -\tan \beta. \quad (11)$$

Let us define  $g$  by

$$g^2 = G'^2 + G^2. \quad (12)$$

Then from (11) and (12) we find

$$G' = -g \sin \beta, \quad G = g \cos \beta. \quad (13)$$

Now we define  $n$  by

$$n^2 = N'^2 + N^2. \quad (14)$$

Then from (8) and (14) we get

$$N' = n \cos \beta, \quad N = n \sin \beta. \quad (15)$$

Now we multiply (13) of Section 4 by  $G$ , (14) of Section 4 by  $G'$  and add, obtaining

$$\frac{1}{2} (G^2 + G'^2)_s - G N' + G' N = 0. \quad (16)$$

By using (13) and (15) in (16) we find

$$g g_s - n g = 0. \quad (17)$$

Hence, if  $g \neq 0$ , we have

$$g_s = n. \quad (18)$$

We now insert (13), (15) and (18) into (17) and (18) of Section 4 and obtain

$$\left( g_{ss} + \frac{T_0}{EI} g \right) g \sin \beta = 0, \quad (19)$$

$$\left( g_{ss} + \frac{T_0}{EI} g \right) g \cos \beta = 0. \quad (20)$$

If  $g \sin \beta \neq 0$  and  $g \cos \beta \neq 0$ , we conclude from (19) and (20) that

$$I = I', \quad (21)$$

$$g_{ss} + \frac{T_0}{EI} g = 0. \quad (22)$$

From (21) and (2) of Section 2, with  $\alpha = \alpha(s)$ ,  $\alpha' = \alpha'(s)$ , it follows that  $\alpha = \alpha'$ . If we set  $\lambda = T_0/\alpha E$ , we may rewrite (22) in the form

$$g_{ss} + \lambda A^{-2} g = 0. \quad (23)$$

From (16) of Section 4 and the definition of  $g$ , we have

$$g(0) = g(L) = 0. \quad (24)$$

If  $\lambda$  denotes the lowest eigenvalue of the problem (23), (24), then  $T_0 = \alpha E \lambda$ . Thus for a fixed  $A^2$ ,  $T_0$  is maximized by maximizing  $\alpha$ . Since  $\alpha = \alpha'$ , the problem of maximizing  $\alpha$  is just the same geometrical problem considered in Section 2. Then since the best value of  $\alpha$  is a constant, independent of  $s$ , the problem of finding  $A(s)$  to maximize  $\lambda$  is just that solved in Section 3. Thus if  $\sin \beta \neq 0$  and  $\cos \beta \neq 0$ , the strongest column is that described toward the end of Section 3.

The exceptional case in which  $\sin \beta \equiv 0$  or  $\cos \beta \equiv 0$  occurs if  $\tau_0 = 0$ , *i.e.*, if the column is untwisted. Then we must compare the lowest eigenvalues for the two cases  $\beta = 0$  and  $\pi/2$ . The smaller of the two is the critical buckling load. The former case yields (22)–(24), and the latter yields the same equation with  $\alpha'$  in place of  $\alpha$ . Thus the critical buckling load is the smaller of  $\alpha E \lambda$  and  $\alpha' E \lambda$ . The determination of the best cross section is thus just that of solving the geometrical problem of Section 2, and then the determination of the best  $A(s)$  is the problem of Section 3. Thus the strongest column is that found in Section 3.

## 6. The nonlinear buckling problem

In the preceding sections we considered the linear buckling problem. That problem was obtained by linearizing the original nonlinear problem around the straight unbuckled state. It yields the surprising result that there is no buckled state unless the load is equal to an eigenvalue. But we expect buckled states to exist for any load exceeding the lowest eigenvalue. In fact, if we consider deflections in a single plane, we expect  $n$  buckled states to exist when the load exceeds the  $n^{\text{th}}$  eigenvalue of the linear problem, but does not exceed the  $(n+1)^{\text{st}}$ . These buckled states correspond to the  $n$  buckling modes associated with the  $n$  eigenvalues less than the load. This expectation is borne out for cylindrical columns for which the explicit solutions of the nonlinear problem (the elastica problem) are known. However for tapered columns, such as those considered here, it has not yet been shown that this is the case. The corresponding statement has been proved to be true by I. I. KOLODNER<sup>1</sup> for a somewhat similar problem.

As a first step toward verifying this conjecture, we shall show that there is a buckled state for any load slightly larger than any buckling load. To do so we shall consider deflections of an untwisted column in a single plane and let  $\vartheta(s)$  denote the angle between the column and a fixed axis in this plane. Then, as follows from the equations (1)–(9) and (16) of Section 4,  $\vartheta(s)$  satisfies the equations

$$(I \vartheta_s)_s + \lambda \sin \vartheta = 0, \quad \vartheta_s(0) = \vartheta_s(L) = 0. \quad (1)$$

The constant  $\lambda = T_0 E^{-1}$  and the positive function  $I(s)$  are given, and  $\vartheta(s)$  is to be found.

It is convenient to introduce the constant  $a$  and the function  $u$  defined by

$$\vartheta = a u, \quad a = \vartheta(0). \quad (2)$$

<sup>1</sup> Comm. Pure Appl. Math. **8**, 395–408 (1955).

Then  $u$  satisfies the equations

$$(I u_s)_s + \lambda a^{-1} \sin(a u) = 0, \quad u(0) = 1, \quad u_s(0) = 0, \quad u_s(L) = 0. \quad (3)$$

If we let  $a$  tend to zero in (3), we obtain the linear problem

$$(I u_s)_s + \lambda u = 0, \quad u(0) = 1, \quad u_s(0) = 0, \quad u_s(L) = 0. \quad (4)$$

This problem has a discrete set of increasing positive eigenvalues  $\lambda_n^0$  and eigenfunctions  $u_n^0$ .

Let us suppose that  $a$  is given and that  $u$  and  $\lambda$  are both to be found. Then we assume that for  $a$  small

$$u = u_n^0 + a^2 u_n^1 + \dots, \quad (5)$$

$$\lambda = \lambda_n^0 + a^2 \lambda_n^1 + \dots. \quad (6)$$

We have expanded in powers of  $a^2$  because (3) is even in  $a$ . Insertion of (5) and (6) into (3) and a simple calculation yield the result

$$\lambda_n^1 = \frac{\lambda_n^0}{6} \frac{\int_0^L (u_n^0)^4 ds}{\int_0^L (u_n^0)^2 ds}. \quad (7)$$

Since  $\lambda_n^1 > 0$ , we can solve (6) for  $a$  in terms of  $\lambda$  provided  $\lambda \geq \lambda_n^0$ . If we denote the solution by  $a_n(\lambda)$ , we have

$$a_n(\lambda) = \left( \frac{\lambda - \lambda_n^0}{\lambda_n^1} \right)^{\frac{1}{2}} [1 + O(\lambda - \lambda_n^0)]. \quad (8)$$

By using this result, it is easy to prove that (3), and therefore (1), has a solution for  $\lambda - \lambda_n^0$  sufficiently small and non-negative. This shows that the nonlinear problem has buckled solutions for any load in a certain half-neighborhood of each buckling load of the linear problem. It also shows that the lowest buckling load of the linear problem is also the lowest buckling load of the nonlinear problem, at least for slightly buckled states. Of course the conjecture concerning the number of buckled states remains to be proved.

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# *An Optimal Poincaré Inequality for Convex Domains*

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## 1. Introduction

Let  $G$  be a convex  $n$ -dimensional domain with boundary  $C$ . It is easily seen that the lowest eigenvalue of the free membrane problem

$$(1.1) \quad \begin{aligned} \Delta v + \mu v &= 0 && \text{in } G \\ \partial v / \partial n &= 0 && \text{on } C \end{aligned}$$

is zero, the eigenfunction being any constant.

This corresponds to the fact that the solution of the interior Neumann problem

$$(1.2) \quad \begin{aligned} \Delta \varphi &= 0 && \text{in } G \\ \partial \varphi / \partial n &\text{ given} && \text{on } C \end{aligned}$$

is only determined to within a constant. The latter is to be fixed by a normalization such as

$$(1.3) \quad \int_G \varphi \, dG = 0.$$

The authors have previously introduced a method for bounding the pointwise value as well as the Dirichlet integral of a solution  $\varphi$  of the exterior Neumann problem in terms of a boundary integral of  $(\partial \varphi / \partial n)^2$  [3]. In order to extend this method to the interior Neumann problem one needs a lower bound for the second eigenvalue  $\mu_2$  of (1). This eigenvalue is characterized by the minimum principle

$$(1.4) \quad \mu_2 = \min_{\int_G u \, dG = 0} \frac{\int_G |\text{grad } u|^2 \, dG}{\int_G u^2 \, dG}.$$

A lower bound for  $\mu_2$  can be used in the interior Neumann problem in the following manner (cf. [3]). Let  $\vec{f}$  be a vector field which is piecewise continuously differentiable throughout  $G$  and points outward on  $C$ , so that

$$(1.5) \quad \vec{f} \cdot \vec{n} \geq k > 0 \quad \text{on } C.$$

For example, if  $G$  is star-shaped with respect to the origin, we may take  $\vec{f}$  to be the radius vector. By the divergence theorem and the inequality  $a^2 + b^2 \geq 2ab$

we have, if  $\varphi$  is normalized by (1.3),

$$\begin{aligned}
 (1.6) \quad \oint_C \varphi^2 \vec{f} \cdot \vec{n} \, dC &= \int_G [\varphi^2 \operatorname{div} \vec{f} + 2\varphi \vec{f} \cdot \operatorname{grad} \varphi] \, dG \\
 &\leq \int_G \varphi^2 [\operatorname{div} \vec{f} + |\vec{f}|^2] \, dG + \int_G |\operatorname{grad} \varphi|^2 \, dG \\
 &\leq [1 + \mu_2^{-1} \max(\operatorname{div} \vec{f} + |\vec{f}|^2)] \int_G |\operatorname{grad} \varphi|^2 \, dG \\
 &= [1 + \mu_2^{-1} \max(\operatorname{div} \vec{f} + |\vec{f}|^2)] \oint_C \varphi \, \partial \varphi / \partial n \, dC.
 \end{aligned}$$

Consequently by SCHWARZ'S inequality

$$(1.7) \quad \oint_C \varphi^2 \vec{f} \cdot \vec{n} \, dC \leq [1 + \mu_2^{-1} \max(\operatorname{div} \vec{f} + |\vec{f}|^2)]^2 \oint_C (\vec{f} \cdot \vec{n})^{-1} (\partial \varphi / \partial n)^2 \, dC.$$

If  $\Gamma$  is a fundamental solution of LAPLACE'S equation with its singularity at the interior point  $P$ , we have

$$\begin{aligned}
 (1.8) \quad |\varphi(P)| &= \left| \oint_C (\Gamma \partial \varphi / \partial n - \varphi \partial \Gamma / \partial n) \, dC \right| \\
 &\leq \left\{ \oint_C (\vec{f} \cdot \vec{n})^{-1} (\partial \varphi / \partial n)^2 \, dC \right\}^{\frac{1}{2}} \left\{ \oint_C \vec{f} \cdot \vec{n} \, \Gamma^2 \, dC \right\}^{\frac{1}{2}} + \\
 &\quad + [1 + \mu_2^{-1} \max(\operatorname{div} \vec{f} + |\vec{f}|^2)] \left\{ \oint_C (\vec{f} \cdot \vec{n})^{-1} (\partial \Gamma / \partial n)^2 \, dC \right\}^{\frac{1}{2}}.
 \end{aligned}$$

Thus, if a lower bound for  $\mu_2$  is known,  $\varphi(P)$  may be explicitly bounded in terms of a the square integral of  $\partial \varphi / \partial n$ .

These results can be extended to general second order differential equations (cf. [3]).

In this paper we shall show that for a convex domain  $G$  in any number of dimensions

$$(1.9) \quad \mu_2 \geq \pi^2 D^{-2}$$

where  $D$  is the diameter of  $G$ . This is the best bound that can be given in terms of the diameter alone in the sense that  $\mu_2 D^2$  tends to  $\pi^2$  for a parallelepiped all but one of whose dimensions shrink to zero.

The inequality (1.9) is in general false for non-convex domains. In fact, for a sequence of domains which tends to two disjoint subdomains,  $\mu_2$  tends to zero. For the special class of domains  $G$  which are symmetric about all the coordinate planes of a rectangular coordinate system and have the property that the intersection of  $G$  with any line in a coordinate direction is simply connected, the authors have previously obtained an inequality of the form (1.9) with  $D$  replaced by the maximum length of intersection of  $G$  with a line in any coordinate direction [4].

A simple upper bound for  $\mu_2$  for any  $n$ -dimensional domain  $G$  in terms of its volume  $V$  is given by the isoperimetric inequality

$$(1.10) \quad \mu_2 \leq p_n^2 K_n^{\frac{2}{n}} V^{-\frac{2}{n}},$$

where  $K_n$  is the volume of the unit  $n$ -sphere and  $p_n$  is the lowest positive root of the equation

$$(1.11) \quad p J'_{\frac{1}{2}n}(p) - (\tfrac{1}{2}n - 1) J_{\frac{1}{2}n}(p) = 0.$$

Equality is attained when  $G$  is a sphere.

For  $n=2$  this inequality was conjectured by KORNHAUSER & STAKGOLD [2] and proved by SZEGÖ [5]. For general  $n$  the proof was given by one of the authors [6].

The eigenvalue  $\mu_2$  itself is of interest in a variety of problems arising in mathematical physics. In two dimensions it is proportional to the square of the cutoff frequency of the lowest  $H$ -mode of a wave guide [2]. In three dimensions it is proportional to the lowest resonant frequency of an acoustic resonator with perfectly rigid walls. It is also inversely proportional to the relaxation time for diffusion in a body with perfectly reflecting boundary.

The proof of the lower bound (1.9) is based upon a lemma concerning a class of Sturm-Liouville systems. This lemma, which is of some interest in itself, is stated and proved in § 2. The inequality (1.9) is proved for two dimensions in § 3 and for higher dimensions in § 4.

## 2. A one-dimensional lemma

In order to prove the lower bound (1.9) we require a somewhat stronger version of its one-dimensional analogue. It is the following lemma.

**Lemma.** *Let  $p(y)$  be a non-negative convex function of  $y$  defined on the interval  $0 \leq y \leq L$ ; then for any piecewise continuously differentiable function  $u(y)$  which satisfies*

$$(2.1) \quad \int_0^L p(y) u(y) dy = 0$$

*it follows that*

$$(2.2) \quad \int_0^L p(y) [u'(y)]^2 dy \geq \pi^2 L^{-2} \int_0^L p(y) [u(y)]^2 dy.$$

**Proof.** We assume for the moment that  $p$  is strictly positive and twice differentiable. Then the function  $v$  which minimizes the quotient

$$(2.3) \quad \frac{\int_0^L p u'^2 dy}{\int_0^L p u^2 dy}$$

among functions  $u$  satisfying (2.1) must satisfy the Sturm-Liouville system [1, p. 348]

$$(2.4) \quad \begin{aligned} [p v']' + \lambda p v &= 0, \\ v'(0) &= v'(L) = 0, \end{aligned}$$

where  $\lambda$  is the minimum value of the quotient (2.3). We divide the equation (2.4) by  $p$ , differentiate with respect to  $y$ , and introduce the new variable

$$(2.5) \quad w = v' p^{\frac{1}{2}}.$$

The function  $w$  satisfies the Sturm-Liouville system

$$(2.6) \quad \begin{aligned} w'' + \left[ \frac{1}{2} \frac{p''}{p} - \frac{3}{4} \frac{p'^2}{p^2} \right] w + \lambda w &= 0, \\ w(0) = w(L) &= 0. \end{aligned}$$

Because of the convexity of  $p$  the term in square brackets is non-positive. Hence, multiplying (2.6) by  $w$  and integrating by parts, we obtain

$$(2.7) \quad \lambda \geq \frac{\int_0^L w'^2 dy}{\int_0^L w^2 dy}.$$

Since  $w(0) = w(L) = 0$  the quotient on the right of (2.7) is bounded below by the first eigenvalue of the vibrating string with fixed ends. Thus

$$(2.8) \quad \lambda \geq \pi^2 L^{-2}.$$

Since  $\lambda$  is the minimum of the quotient (2.3), (2.2) is proved when  $p$  is strictly positive and twice differentiable.

If  $\tilde{u}$  is any function defined on the interval  $0 \leq y \leq L$ , the function

$$(2.9) \quad u(y) \equiv \tilde{u}(y) - \left[ \int_0^L p dy \right]^{-1} \int_0^L p \tilde{u} dy$$

will satisfy (2.4). Hence (2.2) implies

$$(2.10) \quad \int_0^L p \tilde{u}'^2 dy \geq \pi^2 L^{-2} \left\{ \int_0^L p \tilde{u}^2 dy - \left[ \int_0^L p dy \right]^{-1} \left[ \int_0^L p \tilde{u} dy \right]^2 \right\}.$$

Clearly (2.10) is valid for the uniform limit of admissible functions  $p$ . In particular, then,  $p$  may be any non-negative convex function of  $y$ . Thus the lemma is proved.

*Remarks.* 1. The convexity of  $p$  was used only to show that the square bracket in (2.6) is non-positive. For this purpose it is sufficient to assume that  $p^{-\frac{1}{2}}$  is a concave function of  $y$ . Therefore the lemma actually holds under this weaker condition.

2. By the minimax theorem [1, p. 352] we can show that if  $p^{-\frac{1}{2}}$  is a concave function of  $y$ , the eigenvalues of the Sturm-Liouville system (2.4) satisfy the inequality

$$(2.11) \quad \lambda_k \geq (k-1)^2 \pi^2 L^{-2}, \quad k = 1, 2, \dots$$

3. Equality in (2.11) is obtained if and only if  $p^{-\frac{1}{2}}$  is linear in  $y$ . If  $p$  is assumed convex, it must then be constant.

### 3. The two-dimensional case

Let  $G$  be a convex plane domain with boundary  $C$ . Let  $\mu_2$  be defined as the infimum<sup>1</sup> of the quotient

$$(3.1) \quad \frac{\int_G |\text{grad } u|^2 dG}{\int_G u^2 dG}$$

<sup>1</sup> If the boundary  $C$  is smooth so that the problem (1.1) possesses eigenvalues,  $\mu_2$  is the second eigenvalue of (1.1).

among functions which have bounded second derivatives in  $G$  and satisfy

$$(3.2) \quad \int_G u \, dG = 0.$$

Let  $u$  be such a function. Consider the set of lines through the centroid of  $G$ . It follows from continuity that at least one such line divides  $G$  into two convex subdomains of equal area over each of which the integral of  $u$  vanishes. We now divide each of these subdomains into two more convex subdomains of equal area over each of which the integral of  $u$  vanishes.

Continuing this process, we arrive after a finite number of steps at a division of  $G$  into convex subdomains  $G_v$  of arbitrarily small equal areas  $A_v$ . Furthermore,

$$(3.3) \quad \int_{G_v} u \, dG = 0$$

on each  $G_v$ .

Let  $\varrho_v$  be the radius of the largest circle contained in  $G_v$ . Then clearly

$$(3.4) \quad A_v \geq \pi \varrho_v^2.$$

Hence, if  $A_v$  is sufficiently small, the width  $\varrho_v$  of  $G$  is less than a preassigned  $\varepsilon$ :

$$(3.5) \quad \varrho_v \leq \varepsilon.$$

This means that  $G_v$  is contained between two parallel lines at distance  $\varepsilon$ . We introduce a rectangular coordinate system with the  $x_2$ -axis along one of these lines and the  $x_1$ -axis tangent to one end of  $G_v$ . Let  $L_v$  be the length of the projection of  $G_v$  on the  $x_2$ -axis. Clearly,  $L_v \leq D$ . Let  $p(y)$  be the length of the intersection of  $G_v$  with the line  $x_2 = y$ . Then  $p(y) \leq \varepsilon$ . Because of the convexity of  $G_v$ ,  $p(y)$  is convex in  $y$ .

Let  $M$  be a bound for the absolute values of  $u$  and its first and second derivatives. Then by the mean value theorem

$$(3.6) \quad \left| \int_{G_v} \left( \frac{\partial u}{\partial x_2} \right)^2 dG - \int_0^{L_v} p(y) [u(0, y)]^2 dy \right| \leq 2M^2 A_v \varepsilon,$$

$$(3.7) \quad \left| \int_{G_v} u^2 dG - \int_0^{L_v} p(y) [u(0, y)]^2 dy \right| \leq 2M^2 A_v \varepsilon,$$

and

$$(3.8) \quad \left| \int_{G_v} u \, dG - \int_0^{L_v} p(y) u(0, y) dy \right| \leq M A_v \varepsilon.$$

Applying the inequality (2.10) of the lemma, we find, using (3.3) and  $L_v \leq D$ , that

$$(3.9) \quad \begin{aligned} \int_{G_v} |\text{grad } u|^2 dG &\geq \int_{G_v} \left( \frac{\partial u}{\partial x_2} \right)^2 dG \\ &\geq \pi^2 D^{-2} \int_{G_v} u^2 dG - 2M^2 \left( 1 + \pi^2 D^{-2} \left[ 1 + \frac{1}{2} \varepsilon \right] \right) A_v \varepsilon. \end{aligned}$$

We sum these inequalities over all the subdomains  $G_v$ . The sum of the  $A_v$  is the area of  $G$ . Since  $\varepsilon$  is arbitrarily small, we obtain the inequality

$$(3.10) \quad \int_G |\operatorname{grad} u|^2 dG \geq \pi^2 D^{-2} \int_G u^2 dG.$$

Since  $u$  is any function with bounded second derivatives satisfying (3.2), we have, by definition,

$$(3.11) \quad \mu_2 \geq \pi^2 D^{-2}.$$

#### 4. The $n$ -dimensional case

Let  $G$  be a convex  $n$ -dimensional domain with boundary  $C$  ( $n \geq 3$ ). We again define  $\mu_2$  as the infimum of the Rayleigh quotient (3.1) among functions  $u$  having bounded second derivatives in  $G$  and satisfying the conditions (3.2)<sup>1</sup>.

Let  $u$  be such a function. We consider the set of  $n-1$ -planes of the form  $ax_{n-1} + bx_n = c$  passing through the centroid of  $G$ . By continuity we find that at least one of these planes divides  $G$  into two subdomains of equal  $n$ -volumes over each of which the integral of  $u$  vanishes. We divide these subdomains in the same way and continue the process until  $G$  is divided into subdomains  $G_v$  of arbitrarily small  $n$ -volume  $V_v^{[n]}$ . If  $\varrho_v$  is the radius of the largest inscribed  $n$ -sphere, we have

$$(4.1) \quad V_v^{[n]} \geq K_n \varrho_v^n$$

where  $K_n$  is the volume of the unit  $n$ -sphere. Hence, by a sufficiently large number of subdivisions, we can make  $\varrho_v$  less than a preassigned  $\varepsilon$ . This means that each  $G_v$  is contained between two parallel  $n-1$ -planes at distance  $\varepsilon$ . In a particular  $G_v$  we introduce new rectangular coordinates with the  $x_1$ -axis normal to these planes. We proceed to subdivide  $G_v$  by means of planes of the form  $ax_{n-1} + bx_n = c$  into subdomains on each of which the integral of  $u$  vanishes. We make these dividing planes pass through the centroids of the projections on  $x_1=0$  of the domains being divided. After a finite number of such divisions we obtain subdomains  $G'_v$  whose projections on  $x_1=0$  have arbitrarily small  $n-1$  volumes  $V_v^{[n-1]}$ . It follows as before that if  $V_v^{[n-1]}$  is sufficiently small, the projection of  $G'_v$  on  $x_1=0$  lies between two parallel  $n-2$ -planes at distance at most  $\varepsilon$ . We keep the  $x_1$ -direction fixed and choose the  $x_2$ -direction of a new rectangular coordinate system in  $G'_v$  perpendicular to these planes.

If  $n > 3$  we divide each  $G'_v$  further by means of planes  $ax_{n-1} + bx_n = c$  passing through the centroids of the projections on  $x_1 = x_2 = 0$  of  $G'_v$  and of the succeeding domains.

In this way we eventually obtain a subdivision of  $G$  into a finite number of convex subdomains  $G''_v$  over each of which the integral of  $u$  vanishes. Furthermore, each  $G''_v$  is contained in a parallelepiped of the form

$$(4.2) \quad \begin{aligned} 0 &\leq x_i \leq \varepsilon, & i &= 1, 2, \dots, n-1 \\ 0 &\leq x_n \leq L_v \end{aligned}$$

with respect to suitable rectangular coordinates.

<sup>1</sup> If the boundary  $C$  is smooth so that the problem (1.1) possesses eigenvalues,  $\mu_2$  is the second eigenvalue of (1.1).

Let  $\phi(y)$  be the  $n-1$  volume of the intersection of  $G_v''$  with  $x_n=y$ . Then  $\phi(y)$  is convex because of the convexity of  $G_v''$ , and  $\phi(y) \leq \varepsilon^{n-1}$  by (4.2).

The inequality

$$(4.3) \quad \mu_2 \geq \pi^2 D^{-2}$$

is now derived exactly as in § 3.

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# *Error Estimates for the Numerical Solution of Elliptic Differential Equations*

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The numerical treatment of boundary-value problems for elliptic differential equations is often based on the difference method. The solution  $U$  produced by it, which is hoped to be an approximation to the exact solution  $u$ , is defined at the finitely many points of a grid and is obtained as the solution of an algebraic problem approximating the original boundary-value problem. The accuracy of this procedure will depend, of course, upon the fineness of the mesh. The estimation of the deviation  $U - u$  in terms of this fineness is an important task. On the other hand, the quality of the error estimates obtainable finds its restrictions in the regularity properties of the solution or, what is the same, in the behavior of the given data.

In what follows the boundary-value problem

$$(I) \quad \begin{aligned} \mathfrak{L}(u) &\equiv a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} = f(x, y) \quad \text{for } 0 < x, y < 1 \\ u &= 0 \quad \text{for } (x, y) \text{ on the boundary} \end{aligned}$$

is considered. By replacing the derivatives  $u_{xx}$ ,  $u_{xy}$  and  $u_{yy}$  by difference quotients  $\nabla_{xx} U_{i,k}$ ,  $\nabla_{xy} U_{i,k}$  and  $\nabla_{yy} U_{i,k}$  of the grid function  $U = \{U_{i,k}\}$  in a square mesh of width  $h = 1/(N+1)$ , a system of linear equations  $L_{i,k}(U) = F_{i,k}$  is obtained. It is well known that an inequality  $|U - u| \leq CM_4 h^2$  can be derived, as was first shown, in the case  $b(x, y) = 0$ , by S. GERSCHGORIN [3]. Here  $M_4$  denotes a bound for the fourth derivatives of the solution  $u$ . In order, however, that the solution  $u$  be four times differentiable and that such a bound exist at all, one has to assume that the coefficients  $a, b, c$ , and the right-hand side  $f$  possess continuous second derivatives at least. Unfortunately, these assumptions are violated in many applications. Even in cases where  $a, b, c$  are well behaved the function  $f$ , often characterizing an exterior force, will not, in general, be continuous. Then an error estimate involving  $M_4$  is useless. It is the purpose of this paper to derive error estimates under reduced assumptions. The coefficients  $a, b, c$  will be assumed to be continuous. From the continuity of the right-hand side  $f$  it can only be deduced that the second derivatives of  $u$  are square-integrable. But, as a matter of fact, the same statement is true even for square-integrable  $f$ . Therefore,  $f$  will be supposed to be an  $L^2$ -function. Then for the values  $F_{i,k}$  in the difference equations certain means of  $f$  must be taken. In cases where  $f$  corresponds to a point force, to a piecewise constant force, etc., these means can easily be computed.

The result, which in this introduction is given only for the case of a continuous function  $f(x, y)$ , is the following:

$$(II) \quad |U_{i,h} - u_{i,h}| \leq C[\varepsilon_\varrho(f) + \|f\|(\varepsilon_\varrho(a) + \varepsilon_\varrho(b) + \varepsilon_\varrho(c) + h^2\varrho^{-4} + \varrho)].$$

Here the moduli of continuity  $\varepsilon_\varrho$  are defined as  $\varepsilon_\varrho(a) = \text{Max } |a(x_1, y_1) - a(x_2, y_2)|$  for  $0 \leq x_1, x_2, y_1, y_2 \leq 1$  and  $|x_1 - x_2|, |y_1 - y_2| \leq \varrho$ , etc.  $\|f\|$  denotes the  $L^2$ -norm of  $f$ , and  $C$  is a numerical constant depending only on the constants of ellipticity of the differential operator  $\mathfrak{L}(u)$ . For  $\varrho$  any number greater than  $h$  can be taken. With  $\varrho = h^{\frac{2}{3}}$  the last two terms in (II) contract to  $2h^{\frac{2}{3}}$ , and the bracket tends to zero as  $h \rightarrow 0$ .

In the lemmas on which the error estimate (II) is based use is made of the theory of elliptic differential equations, especially of results of L. BERS [1], L. BERS & L. NIRENBERG [2], L. NIRENBERG [5], S. SOBOLEV [6]. The function spaces  $H_\alpha$ , introduced by P. LAX [4], prove to be a helpful tool. As a link connecting the functions  $u$  and  $U$ , repeated mean values of  $u$  over squares with side  $\varrho \geq h$  (for instance  $\varrho = h^{\frac{2}{3}}$ ) are introduced. Mean values over squares with side  $h$  have also been employed by H. WEINBERGER [7] for his study of K. O. FRIEDRICH'S symmetric hyperbolic systems. It seems, however, as indicated also by inequality (II), that the choice  $\varrho = h$  does not give the desired result.

## § 1

The boundary-value problem under consideration is

$$(1) \quad \begin{aligned} \mathfrak{L}(u) &\equiv a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} = f(x, y) \quad \text{for } (x, y) \in S \\ u &= 0 \quad \text{for } (x, y) \text{ on the boundary } \partial S. \end{aligned}$$

Here  $S$  denotes the square  $0 < x, y < 1$ . The coefficients  $a, b, c$  and the right-hand side  $f$  are subjected to the following conditions:

i)  $a, b, c$  are continuous in  $\bar{S} = S + \partial S$ .

ii) There exist two constants  $q$  and  $Q$  ( $0 < q \leq Q$ ) such that

$$(2) \quad q(\xi^2 + \eta^2) \leq a\xi^2 + 2b\xi\eta + c\eta^2 \leq Q(\xi^2 + \eta^2)$$

for all real numbers  $\xi, \eta$  and all  $(x, y) \in \bar{S}$  (uniform ellipticity).

iii)  $f$  is square-integrable over  $S$ .

Under these assumptions, as is proved in the theory of elliptic differential equations, there exists a unique solution  $u(x, y)$  of (1). The second derivatives of  $u$  are square-integrable functions, and the differential equation (1<sub>1</sub>) is satisfied almost everywhere. It is easy to derive estimates for the square norms of the second derivatives of  $u$  in terms of the square norm of the right-hand side  $f$ . In order to do this we observe first that the integral

$$\iint_S (v_{xx} v_{yy} - v_{xy}^2) dx dy$$

vanishes for every function  $v(x, y)$  which is of class  $C^3$  in  $S$  and of class  $C^2$  in  $\bar{S}$  with boundary values zero. This follows directly by partial integration from the identity  $v_{xx} v_{yy} - v_{xy}^2 = (v_x v_{yy})_x - (v_x v_{xy})_y$ . Since  $u$  and its first and second

derivatives can be approximated in the square mean by functions  $v$  with these properties and their derivatives, the same equality holds true for  $u$ :

$$(3) \quad \iint_S (u_{xx} u_{yy} - u_{xy}^2) dx dy = 0.$$

Now multiplying the differential equation (1<sub>1</sub>) by  $u_{xx}/c$ , we obtain

$$(4) \quad \frac{1}{c} [a u_{xx}^2 + 2b u_{xx} u_{xy} + c u_{xy}^2] + (u_{xx} u_{yy} - u_{xy}^2) = \frac{f}{c} u_{xx}.$$

But

$$q \leq c \leq Q,$$

$$q(u_{xx}^2 + u_{xy}^2) \leq a u_{xx}^2 + 2b u_{xx} u_{xy} + c u_{xy}^2 \leq Q(u_{xx}^2 + u_{xy}^2)$$

on account of the uniform ellipticity. So, integrating equation (4), we find that

$$(5) \quad \frac{q}{Q} \iint_S (u_{xx}^2 + u_{xy}^2) dx dy \leq \frac{1}{q} \iint_S |f u_{xx}| dx dy.$$

Introducing the customary norms  $\|f\| = \left( \iint_S f^2 dx dy \right)^{\frac{1}{2}}$  etc. and utilizing the inequality  $|\alpha\beta| \leq \frac{\varepsilon}{2} \alpha^2 + \frac{1}{2\varepsilon} \beta^2$ , valid for real numbers  $\alpha, \beta$  and  $\varepsilon > 0$ , we can write

$$(6) \quad \|u_{xx}\|^2 + \|u_{xy}\|^2 \leq \frac{\varepsilon Q}{2q^2} \|u_{xx}\|^2 + \frac{Q}{2\varepsilon q^2} \|f\|^2.$$

In a similar manner the other inequality

$$(7) \quad \|u_{xy}\|^2 + \|u_{yy}\|^2 \leq \frac{\varepsilon Q}{2q^2} \|u_{yy}\|^2 + \frac{Q}{2\varepsilon q^2} \|f\|^2$$

is obtained. If we set  $\varepsilon = q^2/Q$  and add the two inequalities (6) and (7) (omitting a term  $2\|u_{xy}\|^2$ ), we obtain, as announced, the estimate

$$(8) \quad \|u_{xx}\|^2 + 2\|u_{xy}\|^2 + \|u_{yy}\|^2 \leq \frac{2Q^2}{q^4} \|f\|^2.$$

## § 2

The system of difference equations is obtained in the following way: We cover  $S$  with a square mesh of width  $h = \frac{1}{N+1}$ , so obtaining a grid domain  $S^h$  with the boundary  $\partial S^h$ :

$$S^h: \{P_{ik} = (ih, kh); 1 \leq i, k \leq N\},$$

$$\partial S^h: \{P_{ik} = (ih, kh); i = 0, N+1 \text{ or } k = 0, N+1\}.$$

At all grid points  $P_{ik}$  of  $S^h$  we define the difference quotients of a grid function  $U^h = \{U_{ik}\}$ :

$$\nabla_{xx} U_{ik} = \frac{1}{h^2} (U_{i+1,k} - 2U_{ik} + U_{i-1,k})$$

$$(9) \quad \nabla_{xy} U_{ik} = \frac{1}{4h^2} (U_{i+1,k+1} - U_{i-1,k+1} + U_{i-1,k-1} - U_{i+1,k-1})$$

$$\nabla_{yy} U_{ik} = \frac{1}{h^2} (U_{i,k+1} - 2U_{ik} + U_{i,k-1}).$$

The boundary-value problem (4) is replaced by the algebraic problem

$$(10) \quad \begin{aligned} L_{i,k}(U^h) &\equiv a_{i,k} \nabla_{xx} U_{i,k} + 2b_{i,k} \nabla_{xy} U_{i,k} + c_{i,k} \nabla_{yy} U_{i,k} = F_{i,k} \quad \text{for } P_{i,k} \in S^h \\ U_{i,k} &= 0 \quad \text{for } P_{i,k} \in \partial S^h. \end{aligned}$$

Here  $a_{i,k}$ ,  $b_{i,k}$ ,  $c_{i,k}$  stand for the values of the coefficients  $a$ ,  $b$ ,  $c$  in the grid points. The values of the function  $f(x, y)$  need not, of course, exist at the grid points  $P_{i,k}$ . So, if  $f$  is not continuous (in which case we can take  $F_{i,k} = f(ih, kh)$ ), we shall choose for the  $F_{i,k}$  certain mean values which will be discussed later.

Our next problem is to derive a difference analogue of the *a priori* estimate (8). To this purpose we first observe that there exists a counterpart of equation (3) for any grid function  $V^h$  vanishing on  $\partial S^h$ , namely

$$(11) \quad \sum_{i,k=1}^{N+1} (\tilde{\nabla}_{xy} V_{i,k})^2 = \sum_{i,k=1}^N (\nabla_{xx} V_{i,k}) (\nabla_{yy} V_{i,k}).$$

$\tilde{\nabla}_{xy}$  denotes the unsymmetric mixed difference quotient

$$(12) \quad \tilde{\nabla}_{xy} V_{i,k} = \frac{1}{h^2} (V_{i,k} - V_{i-1,k} + V_{i-1,k-1} - V_{i,k-1}).$$

The difference quotient  $\nabla_{xy} V_{i,k}$  can be expressed as a sum

$$(13) \quad \nabla_{xy} V_{i,k} = \frac{1}{4} (\tilde{\nabla}_{xy} V_{i,k} + \tilde{\nabla}_{xy} V_{i+1,k} + \tilde{\nabla}_{xy} V_{i+1,k+1} + \tilde{\nabla}_{xy} V_{i,k+1}).$$

SCHWARZ'S inequality gives

$$(14) \quad (\nabla_{xy} V_{i,k})^2 \leq \frac{1}{4} [(\tilde{\nabla}_{xy} V_{i,k})^2 + (\tilde{\nabla}_{xy} V_{i+1,k})^2 + (\tilde{\nabla}_{xy} V_{i+1,k+1})^2 + (\tilde{\nabla}_{xy} V_{i,k+1})^2].$$

Combining (11) and (14), we obtain

$$(15) \quad \sum_{i,k=1}^N \{(\nabla_{xx} V_{i,k}) (\nabla_{yy} V_{i,k}) - (\nabla_{xy} V_{i,k})^2\} \geq 0.$$

Now we proceed in the same way as before, obtaining

$$(16) \quad \frac{1}{c_{i,k}} \{a_{i,k} (\nabla_{xx} U_{i,k})^2 + 2b_{i,k} (\nabla_{xx} U_{i,k}) (\nabla_{xy} U_{i,k}) + c_{i,k} (\nabla_{xy} U_{i,k})^2\} + \\ + \{(\nabla_{xx} U_{i,k}) (\nabla_{yy} U_{i,k}) - (\nabla_{xy} U_{i,k})^2\} = \frac{F_{i,k}}{c_{i,k}} (\nabla_{xx} U_{i,k}).$$

It is useful to introduce the following norm for a grid function  $V^h$ :

$$(17) \quad \|V\|_g = \left( h^2 \sum_{i,k=1}^N V_{i,k}^2 \right)^{\frac{1}{2}}.$$

An argument similar to the one following equation (4) yields

$$(18) \quad \|\nabla_{xx} U\|_g^2 + 2 \|\nabla_{xy} U\|_g^2 + \|\nabla_{yy} U\|_g^2 \leq \frac{2Q^2}{q^4} \|F\|_g^2.$$

### § 3

Any real-valued  $L^2$ -function  $v(x, y)$  defined in  $S$  can be characterized by its Fourier coefficients

$$(19) \quad v^{mn} = 4 \iint_S v(x, y) \sin(\pi m x) \sin(\pi n y) dx dy.$$

According to the growth behavior of the Fourier coefficients we introduce classes of functions:

**Definition.** A function  $v$  is said to belong to class  $H_\alpha$ , if the series

$$(20) \quad \frac{1}{4} \sum_{m,n=1}^{\infty} (1+m^2+n^2)^\alpha (v^{mn})^2 \equiv \|v\|_\alpha^2$$

converges.

Clearly, every  $L^2$ -function belongs to class  $H_0$ , and its square norm coincides with the newly defined  $O$ -norm, so that we shall omit the index 0.

If we denote by  $u^{mn}$  the Fourier coefficients of the solution  $u$  of the boundary-value problem (1), then the inequality (8) can be written in the form

$$\frac{1}{4} \sum_{m,n=1}^{\infty} \pi^4 (m^2 + n^2)^2 (u^{mn})^2 \leq \frac{2Q^2}{q^4} \|f\|^2;$$

or, using the fact that  $1+m^2+n^2 \leq \frac{3}{2}(m^2+n^2)$ ,

$$(21) \quad \|u\|_2 \leq \frac{3Q}{\pi^2 q^2 \sqrt{2}} \|f\|.$$

In the following lemmas we shall derive a number of inequalities. For the sake of clarity we shall not always give explicit values for the numerical constants involved but rather denote them by  $C_1$ ,  $C_2$ , etc.

**Lemma 1.** If  $v$  belongs to class  $H_\alpha$  ( $\alpha > 1$ ), then  $v$  is continuous in  $\bar{S}$  and, for  $\varrho \leq \frac{1}{5}$ ,

$$(22) \quad \max_{(x,y) \in \bar{S}} |v(x,y)| \leq C_1(\alpha) \|v\|_\alpha,$$

$$(23) \quad \varepsilon_\varrho(v) \leq \begin{cases} C_2(\alpha) \varrho^{\min(\alpha-1, 1)} \|v\|_\alpha & \text{for } \alpha \neq 2 \\ C_3 \varrho \cdot \left(\log \frac{1}{\varrho}\right)^{\frac{1}{2}} \|v\|_\alpha & \text{for } \alpha = 2. \end{cases}$$

Here  $\varepsilon_\varrho(v)$  denotes the maximum of  $|v(x_1, y_1) - v(x_2, y_2)|$  for  $|x_1 - x_2|$ ,  $|y_1 - y_2| \leq \varrho$  and  $(x_1, y_1), (x_2, y_2) \in \bar{S}$ .

*Proof.* We first observe that the Fourier series  $\sum_{m,n=1}^{\infty} v^{mn} \sin(\pi m x) \sin(\pi n y)$  converges absolutely and uniformly in  $\bar{S}$ . This follows from

$$\begin{aligned} \left( \sum_{m,n=1}^{\infty} |v^{mn}| \right)^2 &\leq \left( \sum_{m,n=1}^{\infty} (1+m^2+n^2)^{-\alpha} \right) \left( \sum_{m,n=1}^{\infty} (1+m^2+n^2)^\alpha (v^{mn})^2 \right) \\ &\leq 2\pi \int_0^\infty \frac{r dr}{(1+r^2)^\alpha} \cdot \|v\|_\alpha^2 = \frac{\pi}{\alpha-1} \|v\|_\alpha^2. \end{aligned}$$

The same inequality yields (22) with  $C_1(\alpha) = \sqrt{\frac{\pi}{\alpha-1}}$ . Further, again applying SCHWARZ's inequality, we see that

$$|v(x_1, y_1) - v(x_2, y_2)|^2 \leq \sigma \|v\|_\alpha^2,$$

$$\sigma = 4 \sum_{m,n=1}^{\infty} (1+m^2+n^2)^{-\alpha} (\sin(\pi m x_1) \sin(\pi n y_1) - \sin(\pi m x_2) \sin(\pi n y_2))^2.$$

Now

$$\begin{aligned} & (\sin(\pi m x_1) \sin(\pi n y_1) - \sin(\pi m x_2) \sin(\pi n y_2))^2 \\ & \leq 2\pi^2(1 + m^2 + n^2) [\text{Max}(|x_1 - x_2|, |y_1 - y_2|)]^2. \end{aligned}$$

Hence, for  $R \geq 5$ ,

$$\sigma \leq 8\pi^2 [\text{Max}(|x_1 - x_2|, |y_1 - y_2|)]^2 \sum_{m^2+n^2 \leq R^2} (1 + m^2 + n^2)^{1-\alpha} + 16 \sum_{m^2+n^2 > R^2} (1 + m^2 + n^2)^{-\alpha}.$$

The first sum can be estimated as follows:

$$\begin{aligned} \sum_{m^2+n^2 \leq R^2} (1 + m^2 + n^2)^{1-\alpha} & \leq \frac{\pi}{2} \int_0^R \frac{r dr}{(1 + r^2)^{\alpha-1}} \\ & = \frac{\pi}{4} \int_1^{1+R^2} \xi^{1-\alpha} d\xi \leq \begin{cases} \frac{\pi}{4(2-\alpha)} (1 + R^2)^{2-\alpha} & \text{for } \alpha < 2 \\ \frac{\pi}{4} \log(1 + R^2) & \text{for } \alpha = 2 \\ \frac{\pi}{4(\alpha-2)} & \text{for } \alpha > 2. \end{cases} \end{aligned}$$

For the second sum we obtain

$$\begin{aligned} 16 \sum_{m^2+n^2 > R^2} (1 + m^2 + n^2)^{-\alpha} & \leq 8\pi \int_{R-\sqrt{2}}^{\infty} \frac{r dr}{(1 + r^2)^{\alpha}} \\ & = \frac{4\pi}{\alpha-1} (1 + (R - \sqrt{2})^2)^{1-\alpha} \leq \frac{4\pi}{\alpha-1} \left(\frac{4}{R^2}\right)^{\alpha-1}. \end{aligned}$$

Assume that  $|x_1 - x_2| \leq \varrho$  and  $|y_1 - y_2| \leq \varrho$ . In the case when  $\alpha > 2$ , let  $R \rightarrow \infty$ . Then

$$\sigma \leq \frac{2\pi^3}{\alpha-2} \varrho^2.$$

In the case when  $\alpha \leq 2$ , put  $R = \frac{1}{\varrho}$ , and make use of the inequalities  $1 + R^2 \leq 2R^2$  and  $\log(1 + R^2) \leq 4 \log R$ . Then

$$\sigma \leq \begin{cases} \varrho^2 |\log \varrho| \left(8\pi^3 + \frac{16\pi}{\log 5}\right) & \text{for } \alpha = 2 \\ \varrho^{2(\alpha-1)} \left(\frac{4\pi^3}{2-\alpha} + \frac{16\pi}{\alpha-1}\right) & \text{for } \alpha < 2. \end{cases}$$

The assertions of Lemma 1 follow from these inequalities.

**Lemma 2.** Let  $v$  belong to class  $H_\alpha$  ( $\alpha > 1$ ). Denote by  $v^N(x, y)$  the finite sum

$$(24) \quad v^N(x, y) = \sum_{m, n=1}^N v^{mn} \sin(\pi m x) \sin(\pi n y),$$

and put  $W_{i,k} = v(ih, kh) - v^N(ih, kh)$ . Then

$$(25) \quad \|W\|_g \leq C_4(\alpha) h^\alpha \|v\|_\alpha.$$

*Proof.* The grid function  $W_{i,k}$  can be expressed in terms of its "discrete" Fourier coefficients

$$(26) \quad W^{mn} = 4h^2 \sum_{i, k=1}^N W_{i,k} \sin(\pi h m i) \sin(\pi h n k)$$

in the form

$$W_{i,h} = \sum_{m,n=1}^N W^{mn} \sin(\pi h m i) \sin(\pi h n k),$$

and

$$(27) \quad \|W\|_g^2 = \frac{1}{4} \sum_{m,n=1}^N (W^{mn})^2.$$

Using the orthogonality relations

$$(28) \quad \sum_{i=1}^N \sin(\pi h m i) \sin(\pi h \mu i) = \begin{cases} \frac{1}{2h} & \text{for } \mu \equiv m(2(N+1)), \\ -\frac{1}{2h} & \text{for } \mu \equiv -m(2(N+1)), \\ 0 & \text{for } \mu \not\equiv \pm m(2(N+1)), \end{cases}$$

which are valid for  $1 \leq m \leq N$  and all  $\mu \geq 1$ , we see that

$$(29) \quad W^{mn} = \sum \pm v^\mu v^\nu.$$

Here  $v^\mu v^\nu$  are the Fourier coefficients of  $v(x, y)$ , and the sum has to be extended over all  $\mu \equiv \pm m(2(N+1))$ ,  $\nu \equiv \pm n(2(N+1))$  with the exclusion of the pair  $\mu = m, \nu = n$ . Then, again applying SCHWARZ's inequality, we see that

$$\begin{aligned} \frac{1}{4} \sum_{m,n=1}^N (W^{mn})^2 &\leq \frac{1}{4} \sum_{m,n=1}^N \left\{ \sum_{\substack{\mu \equiv \pm m \\ \nu \equiv \pm n \\ (\mu, \nu) \neq (m, n)}} (1 + \mu^2 + \nu^2)^{-\alpha} \right\} \left\{ \sum_{\substack{\mu \equiv \pm m \\ \nu \equiv \pm n \\ (\mu, \nu) \neq (m, n)}} (1 + \mu^2 + \nu^2)^\alpha (v^\mu v^\nu)^2 \right\}, \\ &\leq \|v\|_\alpha^2 \text{Max}_{1 \leq m, n \leq N} \left\{ \sum_{\substack{\mu \equiv \pm m \\ \nu \equiv \pm n \\ (\mu, \nu) \neq (m, n)}} (1 + \mu^2 + \nu^2)^{-\alpha} \right\}. \end{aligned}$$

Now, for any  $m$  and  $n$  in  $1 \leq m, n \leq N$  we have

$$\begin{aligned} \sum_{\substack{\mu \equiv \pm m \\ \nu \equiv \pm n \\ (\mu, \nu) \neq (m, n)}} (1 + \mu^2 + \nu^2)^{-\alpha} &\leq \sum_{p+q \geq 1} (1 + (N+1)^2 (p^2 + q^2))^{-\alpha}, \\ &\leq h^{2\alpha} \left\{ 2 \sum_{p=1}^{\infty} \frac{1}{(h^2 + p^2)^\alpha} + \sum_{p,q=1}^{\infty} \frac{1}{(h^2 + p^2 + q^2)^\alpha} \right\}, \\ &\leq h^{2\alpha} \left\{ 2 \left( 1 + \int_1^{\infty} \frac{d\xi}{\xi^{2\alpha}} \right) + 2^{-\alpha} + \frac{\pi}{2} \int_1^{\infty} \frac{r dr}{r^{2\alpha}} \right\}, \\ &= h^{2\alpha} \left\{ 2 + \frac{2}{2\alpha-1} + 2^{-\alpha} + \frac{\pi}{4(\alpha-1)} \right\}, \\ &\leq h^{2\alpha} \left\{ \frac{9}{2} + \frac{\pi}{4(\alpha-1)} \right\}. \end{aligned}$$

Hence, with (27),

$$(30) \quad \|W\|_g^2 \leq \left( \frac{9}{2} + \frac{\pi}{4(\alpha-1)} \right) h^{2\alpha} \|v\|_\alpha^2.$$

q. e. d.

**Lemma 3.** Let  $v$  belong to class  $H_\alpha$  for an integer  $\alpha \geq 2$ . Then the second derivatives  $v_{xx}$ ,  $v_{xy}$ ,  $v_{yy}$  as well as the second difference quotients<sup>1</sup>  $\nabla_{xx} v$ ,  $\nabla_{xy} v$ ,  $\nabla_{yy} v$  belong to class  $H_{\alpha-2}$ , and

$$(31) \quad \|v_{xx}\|_{\alpha-2}, \dots, \|\nabla_{yy} v\|_{\alpha-2} \leq \pi^2 \|v\|_\alpha.$$

*Proof.* The statement follows immediately for  $v_{xx}$  and  $\nabla_{xx} v$  (and  $v_{yy}$  and  $\nabla_{yy} v$ ). Namely, the Fourier coefficients belonging to these functions are

$$-\pi^2 m^2 v^{mn} \quad \text{and} \quad -\frac{\sin^2\left(\frac{\pi h m}{2}\right)}{\left(\frac{\pi h m}{2}\right)^2} \cdot \pi^2 m^2 v^{mn}.$$

The mixed derivative and difference quotient can be represented, in the square norm, by series

$$\pi^2 \sum_{m,n=1}^{\infty} m n v^{mn} \cos(\pi m x) \cos(\pi n y),$$

$$\pi^2 \sum_{m,n=1}^{\infty} \frac{\sin(\pi h m)}{\pi h m} \frac{\sin(\pi h n)}{\pi h n} m n v^{mn} \cos(\pi m x) \cos(\pi n y).$$

Since the norm  $\|v_{xy}\|_{\alpha-2}$  does not change its value if we use the Fourier coefficients with respect to the system  $\{\cos(\pi m x) \cos(\pi n y)\}$ , it follows that

$$\|v_{xy}\|_{\alpha-2}^2 = \frac{\pi^4}{4} \sum_{m,n=1}^{\infty} m^2 n^2 (1 + m^2 + n^2)^{\alpha-2} (v^{mn})^2 \leq \frac{\pi^4}{4} \|v\|_\alpha^2,$$

the estimate for  $\|\nabla_{xy} v\|_{\alpha-2}$  follows similarly. This proves the lemma.

For later purposes we subject our functions to a smoothing process. We define a mean

$$(32) \quad \omega_\varrho(v)(x, y) = \frac{1}{\varrho^2} \int_{x-\frac{\varrho}{2}}^{x+\frac{\varrho}{2}} \int_{y-\frac{\varrho}{2}}^{y+\frac{\varrho}{2}} v(\xi, \eta) d\xi d\eta.$$

In order that this mean be defined for all points of  $\bar{S}$ , we continue the function  $v$  across the sides of  $S$  periodically as an odd function. The relation between the Fourier coefficients of  $v$  and those of  $\omega_\varrho(v)$  is

$$(33) \quad \omega_\varrho(v)^{mn} = \frac{\sin \frac{\pi m \varrho}{2}}{\frac{\pi m \varrho}{2}} \frac{\sin \frac{\pi n \varrho}{2}}{\frac{\pi n \varrho}{2}} v^{mn}$$

so that, in particular,  $|\omega_\varrho(v)^{mn}| \leq |v^{mn}|$  or

$$(34) \quad \|\omega_\varrho(v)\|_\alpha \leq \|v\|_\alpha.$$

**Lemma 4.** Let  $v$  belong to class  $H_\alpha$  ( $\alpha \geq 0$ ). Then  $\omega_\varrho(v)$  belongs to class  $H_{\alpha+1}$ , and

$$(35) \quad \|\omega_\varrho(v)\|_{\alpha+1} \leq \frac{4\sqrt{3}}{\pi^2 \varrho^2} \|v\|_\alpha.$$

<sup>1</sup> They are defined, similarly to (9), by  $\nabla_{xx} v(x, y) = \frac{1}{h^2} [v(x+h, y) - 2v(x, y) + v(x-h, y)]$  etc.

*Proof.* On account of the inequality  $m^2 n^2 \geq \frac{1}{3}(1 + m^2 + n^2)$  we have

$$(1 + m^2 + n^2) (\omega_\varrho(v)^{mn})^2 \leq \frac{48}{\pi^4 \varrho^4} (v^{mn})^2$$

and hence (35). q. e. d.

**Lemma 5.** Let  $v$  belong to class  $H_\alpha$  ( $1 < \alpha < 3$ ). Then, for  $\varrho \leq \frac{1}{5}$ ,

$$(36) \quad \max_{(x, y) \in \bar{S}} |v(x, y) - \omega_\varrho(v)(x, y)| \leq C_5(\alpha) \varrho^{\frac{2(\alpha-1)}{\max(\alpha, 2)}} \|v\|_\alpha.$$

*Proof.* We have

$$v(x, y) - \omega_\varrho(v)(x, y) = \sum_{m, n=1}^{\infty} g_{mn} v^{mn} \sin(\pi m x) \sin(\pi n y)$$

where

$$g_{mn} = 1 - 4 \frac{\sin \frac{\pi m \varrho}{2} \sin \frac{\pi n \varrho}{2}}{\pi^2 m n \varrho^2}.$$

Hence

$$0 \leq g_{mn} \leq \min \left\{ 2, \frac{\pi^2 \varrho^2}{24} (1 + m^2 + n^2) \right\}.$$

Then

$$|v(x, y) - \omega_\varrho(v)(x, y)| \leq 4 \|v\|_\alpha^2 \sum_{m, n=1}^{\infty} g_{mn}^2 (1 + m^2 + n^2)^{-\alpha}.$$

If  $R \geq 4$ , the sum can be estimated as follows:

$$\begin{aligned} \sum_{m, n=1}^{\infty} g_{mn}^2 (1 + m^2 + n^2)^{-\alpha} &= \sum_{m^2 + n^2 \leq R^2} + \sum_{m^2 + n^2 > R^2} \\ &\leq \frac{\pi^4 \varrho^4}{576} \sum_{m^2 + n^2 \leq R^2} (1 + m^2 + n^2)^{2-\alpha} + 4 \sum_{m^2 + n^2 > R^2} (1 + m^2 + n^2)^{-\alpha}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{m^2 + n^2 \leq R^2} (1 + m^2 + n^2)^{2-\alpha} &\leq \sum_{m^2 + n^2 \leq R^2} (1 + m^2 + n^2)^{2-\min(2, \alpha)} \leq \frac{\pi}{2} \int_0^{R+\sqrt{2}} r (1 + r^2)^{2-\min(2, \alpha)} dr \\ &\leq \frac{\pi}{4} \int_1^{2R^2} \xi^{2-\min(2, \alpha)} d\xi \leq \frac{\pi}{4} (2R^2)^{3-\min(2, \alpha)}. \end{aligned}$$

Further

$$\sum_{m^2 + n^2 > R^2} (1 + m^2 + n^2)^{-\alpha} \leq \frac{\pi}{2} \int_{R-\sqrt{2}}^{\infty} \frac{r dr}{(1 + r^2)^\alpha} \leq \frac{\pi}{4} \int_{\frac{1}{2}R^2}^{\infty} \xi^{-\alpha} d\xi = \frac{\pi}{4(\alpha-1)} \frac{3^{\alpha-1}}{R^{2(\alpha-1)}}.$$

Consequently

$$\sum_{m, n=1}^{\infty} g_{mn}^2 (1 + m^2 + n^2)^{-\alpha} \leq \frac{\pi^5 \varrho^4}{576} R^{2(3-\min(\alpha, 2))} + \frac{\pi}{\alpha-1} \frac{3^{\alpha-1}}{R^{2(\alpha-1)}}.$$

Putting  $R = \varrho^{-\frac{2}{\max(2, \alpha)}}$ , we obtain

$$\sum_{m, n=1}^{\infty} g_{mn}^2 (1 + m^2 + n^2)^{-\alpha} \leq \varrho^{\frac{4(\alpha-1)}{\max(2, \alpha)}} \left\{ \frac{\pi^5}{576} + \frac{9\pi}{\alpha-1} \right\},$$

which proves (36).

## § 4

The solution  $u$  of the boundary-value problem (1) belongs to class  $H_2$ . Its second derivatives and difference quotients belong to class  $H_0$ , and in general the corresponding Fourier series will not converge everywhere, in particular not necessarily at the grid points. This difficulty can be overcome by applying the smoothing process of the previous section. We introduce the function

$$(37) \quad \tilde{u}(x, y) = \omega_e^2(u)(x, y) \equiv \omega_e(\omega_e(u))(x, y).$$

$\tilde{u}(x, y)$  belongs to class  $H_4$  according to Lemma 4, and

$$(38) \quad \|\tilde{u}\|_4 \leq C_6 \varrho^{-4} \|f\|$$

$$\left( C_6 = \frac{144 Q}{\pi^6 Q^2 \sqrt{2}} \right).$$

**Lemma 6.** *Let  $D^2$  be any second partial derivative and  $V^2$  the corresponding difference quotient. Then*

$$(39) \quad \|D^2 \tilde{u} - V^2 \tilde{u}\|_g \leq C_8 h^2 \varrho^{-4} \|f\|.$$

The grid norm in (39) is taken for the grid function  $D^2 \tilde{u}(ih, kh) - V^2 \tilde{u}(ih, kh)$ .  $\tilde{u}^N$  is defined as in Lemma 2.

*Proof.* According to Lemma 2 we have ( $\alpha=2$ )

$$\|D^2 \tilde{u} - D^2 \tilde{u}^N\|_g \leq C_4(2) h^2 \|D^2 \tilde{u}\|_2,$$

$$\|V^2 \tilde{u} - V^2 \tilde{u}^N\|_g \leq C_4(2) h^2 \|D^2 \tilde{u}\|_2,$$

and hence, by Lemma 3,

$$(40) \quad \|D^2 \tilde{u} - D^2 \tilde{u}^N\|_g, \quad \|V^2 \tilde{u} - V^2 \tilde{u}^N\|_g \leq C_7 h^2 \varrho^{-4} \|f\|$$

( $C_7 = C_4(2) \pi^2 C_6$ ). Further

$$\|D^2 \tilde{u}^N - V^2 \tilde{u}^N\|_g^2 \leq \frac{1}{4} \sum_{m,n=1}^N g_{mn}^2 (\tilde{u}^{mn})^2.$$

Here

$$g_{mn} = \begin{cases} \pi^2 m^2 \left( 1 - \frac{4 \sin^2 \left( \frac{\pi h m}{2} \right)}{\pi^2 h^2 m^2} \right) & \text{if } D^2 = \partial_{xx}, \\ \pi^2 m n \left( 1 - \frac{\sin(\pi h m) \sin(\pi h n)}{\pi^2 h^2 m n} \right) & \text{if } D^2 = \partial_{xy}, \\ \pi^2 n^2 \left( 1 - \frac{4 \sin^2 \left( \frac{\pi h n}{2} \right)}{\pi^2 h^2 n^2} \right) & \text{if } D^2 = \partial_{yy}. \end{cases}$$

For all these  $g_{mn}$  the inequality

$$g_{mn} \leq \frac{\pi^4 h^2}{12} (1 + m^2 + n^2)^2$$

holds true. Therefore

$$\sum_{m,n=1}^N g_{mn}^2 (\tilde{u}^{mn})^2 \leq \frac{\pi^8 h^4}{144} \sum_{m,n=1}^N (1 + m^2 + n^2)^4 (\tilde{u}^{mn})^2$$

or, using (38),

$$(41) \quad \|D^2 \tilde{u}^N - \nabla^2 \tilde{u}^N\|_g \leq \frac{\pi^4 h^2}{12} \|\tilde{u}\|_4 \leq \frac{\pi^4 C_6}{12} h^2 \varrho^{-4} \|f\|.$$

Combining the inequalities (40) and (41), we obtain

$$\begin{aligned} \|D^2 \tilde{u} - \nabla^2 \tilde{u}\|_g &\leq \|D^2 \tilde{u} - D^2 \tilde{u}^N\|_g + \|D^2 \tilde{u}^N - \nabla^2 \tilde{u}^N\|_g + \|\nabla^2 \tilde{u} - \nabla^2 \tilde{u}^N\|_g \\ &\leq h^2 \varrho^{-4} \|f\| \left( 2C_7 + \frac{\pi^4 C_6}{12} \right) \equiv C_8 h^2 \varrho^{-4} \|f\|, \end{aligned}$$

that is, our assertion (39).

Let  $a(x, y)$  be a continuous function in  $\bar{S}$ . Denote by  $\varepsilon_\tau(a)$  its module of continuity:  $\varepsilon_\tau(a) = \text{Max } |a(x_1, y_1) - a(x_2, y_2)|$ , where the maximum is to be taken over all pairs of points  $(x_1, y_1), (x_2, y_2) \in \bar{S}$  with  $|x_1 - x_2|, |y_1 - y_2| \leq \tau$ . It is easy to extend  $a(x, y)$  into the whole  $(x, y)$ -plane in such a way that the module of continuity is not increased. In the following lemma it will be assumed that  $a(x, y)$  is so extended.

**Lemma 7.** *Let  $a(x, y)$  be continuous, and let  $v$  belong to class  $H_0$ . Then*

$$(42) \quad \|a \omega_\varrho^2(v) - \omega_\varrho^2(av)\|_g \leq 4 \varepsilon_\varrho(a) \|v\|.$$

*Proof.* We have

$$\begin{aligned} &\{a \omega_\varrho^2(v) - \omega_\varrho^2(av)\}(x, y) \\ &= \frac{1}{\varrho^4} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} d\xi d\eta \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} d\xi' d\eta' \{a(x, y) - a(x + \xi + \xi', y + \eta + \eta')\} \times \\ &\quad \times v(x + \xi + \xi', y + \eta + \eta'). \end{aligned}$$

Applying SCHWARZ'S inequality to the inner integral, we obtain

$$|\{a \omega_\varrho^2(v) - \omega_\varrho^2(av)\}(x, y)| \leq \varepsilon_\varrho(a) \left[ \frac{1}{\varrho^2} \int_{x-\varrho}^{x+\varrho} \int_{y-\varrho}^{y+\varrho} v^2(\xi, \eta) d\xi d\eta \right]^{\frac{1}{2}}.$$

Hence, using the fact that  $\varrho \leq ([\varrho/h] + 1)h$ , for  $\varrho \geq h$ ,

$$\begin{aligned} \|a \omega_\varrho^2(v) - \omega_\varrho^2(av)\|_g^2 &\leq h^2 (\varepsilon_\varrho(a))^2 \frac{1}{\varrho^2} 4 \left( \left[ \frac{\varrho}{h} \right] + 1 \right)^2 \iint_S v^2 d\xi d\eta \\ &\leq 16 (\varepsilon_\varrho(a))^2 \|v\|^2. \end{aligned}$$

This proves Lemma 7.

After these preparations we can turn to our estimation of the error. We have

$$(43) \quad L_{i,k}(\tilde{u}) = [\mathfrak{L}(\tilde{u})]_{(i,h,k,h)} + R_{i,k}^{(1)}$$

with

$$(44) \quad R_{i,k}^{(1)} = \{a_{i,k}(\nabla_{xx} - \partial_{xx}) + 2b_{i,k}(\nabla_{xy} - \partial_{xy}) + c_{i,k}(\nabla_{yy} - \partial_{yy})\} \tilde{u}(i, h, k, h).$$

But, on account of the definition (37) of  $\tilde{u}$ ,

$$(45) \quad [\mathfrak{L}(\tilde{u})]_{(i,h,k,h)} = [\omega_\varrho^2(f)]_{i,h,k,h} + R_{i,k}^{(2)}$$

where

$$(46) \quad R_{i,k}^{(2)} = \{ (a \omega_{\varrho}^2(u_{xx}) - \omega_{\varrho}^2(a u_{xx})) + 2b(\omega_{\varrho}^2(u_{xy}) - \omega_{\varrho}^2(b u_{xy})) + \\ + (c \omega_{\varrho}^2(u_{yy}) - \omega_{\varrho}^2(c u_{yy})) \}_{(i,k,h)}.$$

According to Lemmas 6 and 7 and the inequality (8) (without omission), we get

$$(47) \quad \|R^{(1)}\|_g \leq \text{Max}(|a|, |b|, |c|) \{ \|(\nabla_{xx} - \partial_{xx}) \tilde{u}\|_g + 2 \|(\nabla_{xy} - \partial_{xy}) \tilde{u}\|_g + \|(\nabla_{yy} - \partial_{yy}) \tilde{u}\|_g \} \\ \leq 4Q C_8 h^2 \varrho^{-4} \|f\|,$$

$$(48) \quad \|R^{(2)}\|_g \leq 4\varepsilon_{\varrho}(a, b, c) \{ \|u_{xx}\| + 2 \|u_{xy}\| + \|u_{yy}\| \} \\ \leq \frac{4Q\sqrt{6}}{Q^2} \varepsilon_{\varrho}(a, b, c) \|f\|.$$

Here  $\varepsilon_{\varrho}(a, b, c) = \text{Max}(\varepsilon_{\varrho}(a), \varepsilon_{\varrho}(b), \varepsilon_{\varrho}(c))$ .

We compare the solution  $\{U_{i,k}\}$  of the difference equations (40) with the values of the function  $\tilde{u}$  at the grid points, because their difference satisfies the linear equations

$$(49) \quad L_{i,k}(\tilde{u} - U) = \omega_{\varrho}^2(f)(ih, kh) - F_{i,k} + R_{i,k}^{(1)} + R_{i,k}^{(2)}.$$

Consequently, by virtue of inequality (18), we obtain the *main inequality*

$$(50) \quad \|\nabla_{xx}(\tilde{u} - U)\|_g^2 + 2\|\nabla_{xy}(\tilde{u} - U)\|_g^2 + \|\nabla_{yy}(\tilde{u} - U)\|_g^2 \\ \leq C_9 \|\omega_{\varrho}^2(f) - F\|_g^2 + C_{10} \|f\|^2 (\varepsilon_{\varrho}(a, b, c))^2 + C_{11} h^4 \varrho^{-8} \|f\|^2.$$

Here

$$C_9 = \frac{6Q^2}{Q^4}, \quad C_{10} = \frac{576Q^4}{Q^8}, \quad C_{11} = \frac{96Q^4 C_8^2}{Q^4}.$$

The final exploitation of this main inequality (50) depends upon the properties of the function  $f(x, y)$ . We distinguish two cases: If  $f(x, y)$  is continuous in  $\bar{S}$ , then the numerical calculation is carried out with the values  $F_{i,k} = f(ih, kh)$  for the right-hand side of the difference equations. Using Lemma 7 with  $a=f, v=1$ , we then find

$$\|\omega_{\varrho}^2(f) - F\|_g \leq 4\varepsilon_{\varrho}(f).$$

Thus we obtain

$$(51) \quad \|\nabla_{xx}(\tilde{u} - U)\|_g^2 + 2\|\nabla_{xy}(\tilde{u} - U)\|_g^2 + \|\nabla_{yy}(\tilde{u} - U)\|_g^2 \\ \leq \|f\|^2 (C_{10}(\varepsilon_{\varrho}(a, b, c))^2 + C_{11} h^4 \varrho^{-8}) + C_{12} (\varepsilon_{\varrho}(f))^2 \\ (C_{12} = 16C_9).$$

In case  $f$  is not continuous, we must keep the expression  $\frac{1}{4} \|\omega_{\varrho}^2(f) - F\|_g$  in place of  $\varepsilon_{\varrho}(f)$ . Of course, if for the numerical calculation we use the mean values  $\omega_{\varrho}^2(f)(ih, kh)$  for the right-hand side  $F_{i,k}$ , then this term does not appear.

## § 5

The main inequality (50) of the last paragraph is now used to derive the estimate for the difference  $u(ih, kh) - U_{i,k}$ . First we prove

**Lemma 8<sup>2</sup>.** *Let  $W_{i,k}$  be a grid function with vanishing boundary values. Then*

$$(52) \quad \text{Max}_{1 \leq i, k \leq N} |W_{i,k}| \leq \frac{1}{4} \sqrt{1 + \pi} \|(\nabla_{xx} + \nabla_{yy}) W\|_g.$$

<sup>2</sup> This is a finite difference analogy to an inequality of S. SOBOLEV.

*Proof.* As usual, we express  $W_{i,k}$  in terms of its discrete Fourier coefficients

$$W_{i,k} = \sum_{m,n=1}^N W^{mn} \sin(\pi h m i) \sin(\pi h n k).$$

Then we find

$$\|(\nabla_{xx} + \nabla_{yy}) W\|_g^2 = \frac{4}{h^4} \sum_{m,n=1}^N \left( \sin^2\left(\frac{\pi h m}{2}\right) + \sin^2\left(\frac{\pi h n}{2}\right) \right)^2 (W^{mn})^2.$$

On the other hand

$$\begin{aligned} |W_{i,k}|^2 &\leq \left( \sum_{m,n=1}^N |W^{mn}| \right)^2 \leq \left( \sum_{m,n=1}^N \frac{1}{(m^2 + n^2)^2} \right) \left( \sum_{m,n=1}^N (m^2 + n^2)^2 (W^{mn})^2 \right) \\ &\leq \frac{1}{4} \|(\nabla_{xx} + \nabla_{yy}) W\|_g^2 \left( \sum_{m,n=1}^{\infty} \frac{1}{(m^2 + n^2)^2} \right) \\ &\leq \frac{1+\pi}{16} \|(\nabla_{xx} + \nabla_{yy}) W\|_g^2 \end{aligned}$$

since

$$\sum_{m,n=1}^{\infty} \frac{1}{(m^2 + n^2)^2} \leq \frac{1}{4} + \frac{\pi}{2} \int_1^{\infty} \frac{r dr}{r^4} = \frac{1+\pi}{4}.$$

This proves Lemma 8.

The difference  $u(ih, kh) - U_{i,k}$  can be written as

$$[u - \omega_{\varrho}(u)]_{(ih,kh)} + [\omega_{\varrho}(u) - \omega_{\varrho}^2(u)]_{ih,kh} + [\tilde{u}(ih, kh) - U_{i,k}].$$

Applying Lemma 5 for  $\alpha=2$  and using (34), we obtain

$$\begin{aligned} \max_{(x,y) \in \bar{S}} |u - \omega_{\varrho}(u)| &\leq C_5(2) \varrho \|u\|_2, \\ \max_{(x,y) \in \bar{S}} |\omega_{\varrho}(u) - \omega_{\varrho}^2(u)| &\leq C_5(2) \varrho \|\omega_{\varrho}(u)\|_2 \leq C_5(2) \varrho \|u\|_2. \end{aligned}$$

Therefore, Lemma 8 in conjunction with the main inequality (50) gives

$$(53) \quad \begin{aligned} \max_{1 \leq i, k \leq N} |u(ih, kh) - U_{i,k}| \\ \leq C_{13} \|\omega_{\varrho}^2(f) - F\|_g + \|f\| \{C_{14} \varepsilon_{\varrho}(a, b, c) + C_{15} h^2 \varrho^{-4} + C_{16} \varrho\}. \end{aligned}$$

Here

$$\begin{aligned} C_{13} &= \frac{1}{4} \sqrt{2(1+\pi)} C_9, & C_{14} &= \frac{1}{4} \sqrt{2(1+\pi)} C_{10}, \\ C_{15} &= \frac{1}{4} \sqrt{2(1+\pi)} C_{11}, & C_{16} &= \frac{3Q\sqrt{2}}{\pi^2 q^2} C_5(2). \end{aligned}$$

In particular, if we set  $\varrho = h^{\frac{2}{3}}$ ,

$$(54) \quad \begin{aligned} \max_{1 \leq i, k \leq N} |u(ih, kh) - U_{i,k}| \\ \leq C_{13} \|\omega_{h^{\frac{2}{3}}}^2(f) - F\|_g + C_{14} \varepsilon_{h^{\frac{2}{3}}}(a, b, c) \|f\| + (C_{15} + C_{16}) h^{\frac{2}{3}} \|f\|. \end{aligned}$$

As before, the term  $\|\omega_{h^{\frac{2}{3}}}^2(f) - F\|_g$  can be replaced by  $4\varepsilon_{h^{\frac{2}{3}}}(f)$  if the function  $f$  is continuous.

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# *Fehlerabschätzung für die numerische Berechnung von Integralen, die Lösungen elliptischer Differentialgleichungen enthalten*

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*Vorgelegt von C. MÜLLER*

Vorgegeben sei das Randwertproblem einer elliptischen Differentialgleichung

$$(1) \quad \begin{aligned} \mathfrak{L}(u) &\equiv a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} = f(x, y) \quad \text{in } S: \{0 < x, y < 1\} \\ u &= 0 \quad \text{auf dem Rande } \partial S. \end{aligned}$$

Zur numerischen Integration wird (1) durch ein System von Differenzengleichungen für eine in den Gitterpunkten eines quadratischen Netzes mit der Maschenweite  $h = \frac{1}{N+1}$  definierte Gitterfunktion  $U^h = \{U_{i,k}\}$  ersetzt. In den Anwendungen treten häufig Integrale der Form

$$(2) \quad \iint_S p(x, y) u(x, y) dx dy, \dots, \quad \iint_S p(x, y) u_{xx}(x, y) dx dy, \dots$$

auf. Bei der numerischen Berechnung wird man diese Integrale durch Summen

$$(3) \quad h^2 \sum_{i,k=1}^N P_{i,k} U_{i,k}, \dots, \quad h^2 \sum_{i,k=1}^N P_{i,k} V_{xx} U_{i,k}, \dots$$

ersetzen. Es ist das Ziel der vorliegenden Arbeit, Abschätzungen für die dabei resultierenden Fehler zu geben. Eine Abschätzung für die Differenz  $U_{i,k} - u(ih, kh)$  ist bereits in einer vorangehenden Arbeit<sup>1</sup> hergeleitet worden. Dabei war Wert darauf gelegt, die Voraussetzungen für die Koeffizienten  $a, b, c$  und die rechte Seite  $f$  — welche wir hier übernehmen — möglichst gering zu halten:  $a, b$  und  $c$  werden in  $\bar{S} = S + \partial S$  stetig und  $f$  in  $S$  quadratisch integrierbar angenommen. Die Differentialgleichung soll gleichmäßig elliptisch sein.

Im Einklang damit wird von der Funktion  $p(x, y)$  vorausgesetzt, daß sie in  $S$  quadratisch integrierbar ist. Im allgemeinen braucht dann  $p$  in den Gitterpunkten nicht definiert zu sein. Für die in den Summen (3) auftretenden Werte

<sup>1</sup> Error Estimates for the Numerical Solution of Elliptic Differential Equations, vorangehend in dieser Zeitschrift; im folgenden mit  $\mathfrak{N}$  abgekürzt.

$P_{i,k}$  werden vielmehr gewisse Mittelwerte eingesetzt, nämlich

$$(4.1) \quad P_{i,k} = \omega_{\varrho}^2(p)_{i,k} = \frac{1}{\varrho^4} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} p(ih + \xi + \xi', kh + \eta + \eta') d\xi d\eta d\xi' d\eta'.$$

Der Parameter  $\varrho$  kann dabei geeignet gewählt werden. Falls  $p(x, y)$  stetig ist, wird man natürlich

$$(4.2) \quad P_{i,k} = p(ih, kh)$$

nehmen. Wir werden die Fehlerabschätzung erst für den Fall (4.1) durchführen (Ungleichung (30)). In § 5 wird dann die auf (4.2) beruhende modifizierte Fehlerabschätzung angegeben (Ungleichung (32)). Wir beschäftigen uns, was offensichtlich genügt, nur mit den Integralen über zweite Ableitungen von  $u$ .

Nehmen wir für einen Augenblick an, daß  $p$  und  $f$  stetig sind, so läßt sich das Resultat folgendermaßen ausdrücken:

$$(5) \quad \left| \int_S p(x, y) D^2 u dx dy - h^2 \sum_{i,k=1}^N p(ih, kh) \nabla^2 U_{i,k} \right| \leq C \cdot [\|p\| \varepsilon_{\varrho}(f) + \|f\| \varepsilon_{\varrho}(p) + \varepsilon_{\varrho}(p) \varepsilon_{\varrho}(f) + \|p\| \|f\| (\varepsilon_{\varrho}(a, b, c) + h^2 \varrho^{-4})].$$

$D^2 u$  ist eine zweite Ableitung von  $u$  und  $\nabla^2 U$  der entsprechende zweite Differenzenquotient der Gitterfunktion  $U^h$ .  $\| \cdot \|$  bedeutet die  $L^2$ -Norm, und die  $\varepsilon_{\varrho}$  sind die Stetigkeitsmoduln. Indem man  $\varrho = h^{\gamma}$  ( $\gamma < \frac{1}{2}$ ) setzt, strebt die rechte Seite von (5) mit  $h$  nach Null. Falls alle auftretenden Funktionen einer Lipschitz-Bedingung genügen und man  $\gamma = \frac{2}{5}$  annimmt, so läßt sich der Fehler durch die Schranke  $C h^{\frac{2}{5}}$  abschätzen.

In den nachfolgenden Ausführungen werden die Bezeichnungen der Arbeit  $\mathfrak{R}$  beibehalten und die dort gewonnenen Resultate benutzt.

## § 1

Um die späteren Entwicklungen nicht unterbrechen zu müssen, soll der folgende Hilfssatz vorausgeschickt werden.

**Hilfssatz.** Die Funktion  $v(x, y)$  sei in  $S$  quadratisch integrierbar, und es werde  $V(x, y) = \omega_{\varrho}^2(v)(x, y)$  gesetzt ( $\varrho \geq h$ ). Dann gilt

$$(6) \quad \|V\|_{\mathfrak{g}} \leq 4 \|v\|.$$

*Beweis.* Es ist

$$\|V\|_{\mathfrak{g}}^2 = h^2 \sum_{i,k=1}^N \left\{ \frac{1}{\varrho^4} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} d\xi d\eta \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} d\xi' d\eta' v(ih + \xi + \xi', kh + \eta + \eta') \right\}^2.$$

Die Zahl  $\varrho' = h \left( \left\lceil \frac{\varrho}{h} \right\rceil + 1 \right)$  ist größer als  $\varrho$ , und folglich

$$\left\{ \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} d\xi' d\eta' v(ih + \xi + \xi', kh + \eta + \eta') \right\}^2 \leq \varrho^2 \int_{-\varrho'}^{\varrho'} \int_{-\varrho'}^{\varrho'} d\xi' d\eta' v^2(ih + \xi', kh + \eta').$$

Damit wird

$$\|V\|_g^2 \leq h^2 \varrho^{-2} \sum_{i,k=1}^N \int_{-\varrho'}^{\varrho'} \int d\xi' d\eta' v^2(ih + \xi', kh + \eta') \leq 4h^2 \varrho^{-2} \left( \left\lfloor \frac{\varrho}{h} \right\rfloor + 1 \right)^2 \|v\|^2.$$

Es ist aber  $\left\lfloor \frac{\varrho}{h} \right\rfloor + 1 \leq \frac{2\varrho}{h}$ , woraus Ungleichung (6) folgt.

## § 2

Zur Herleitung der Abschätzung (30) bedienen wir uns wie früher der aus der Lösung  $u(x, y)$  des Randwertproblems durch zweimalige Anwendung des Glättungsprozesses  $\omega_\varrho$  gewonnenen Funktion  $\tilde{u}(x, y)$ . Der Ausdruck  $h^2 \sum_{i,k=1}^N P_{i,k} \nabla^2 U_{i,k}$ , welcher sich unter Einführung des skalaren Gitterproduktes

$$(7) \quad (V, W)_g = h^2 \sum_{i,k=1}^N V_{i,k} W_{i,k}$$

abkürzend in der Form  $(P, \nabla^2 U)_g$  schreiben läßt, wird in der folgenden Weise umgeformt:

$$(8) \quad (P, \nabla^2 U)_g = (P, \nabla^2 (U - \tilde{u}))_g + (P, (\nabla^2 - D^2) \tilde{u})_g + (P, D^2 \tilde{u})_g.$$

Zufolge Lemma 6 von  $\mathfrak{N}$  gilt

$$\|(\nabla^2 - D^2) \tilde{u}\|_g \leq C_8 h^2 \varrho^{-4} \|f\|,$$

so daß

$$(9) \quad |(P, (\nabla^2 - D^2) \tilde{u})_g| \leq \|P\|_g \|(\nabla^2 - D^2) \tilde{u}\|_g \leq 4C_8 h^2 \varrho^{-4} \|p\| \|f\|.$$

Bei der letzten Ungleichung ist der in § 1 bewiesene Hilfssatz benutzt.

Für das erste Glied auf der rechten Seite von (8) erhalten wir mit Hilfe der Hauptungleichung ( $\mathfrak{N}$  50)

$$|(P, \nabla^2 (U - \tilde{u}))_g| \leq \|P\|_g \{ \sqrt{C_9} \|\omega_\varrho^2(f) - F\|_g + \sqrt{C_{10}} \|f\| \varepsilon_\varrho(a, b, c) + \sqrt{C_{11}} h^2 \varrho^{-4} \|f\| \},$$

bzw. mit  $\gamma_1 = 4 \text{Max}(\sqrt{C_9}, \sqrt{C_{10}}, \sqrt{C_{11}})$ :

$$(10) \quad |(P, \nabla^2 (U - \tilde{u}))_g| \leq \gamma_1 \|p\| \{ \|\omega_\varrho^2(f) - F\|_g + \|f\| (\varepsilon_\varrho(a, b, c) + h^2 \varrho^{-4}) \}.$$

## § 3

Der dritte Ausdruck rechts in (8) läßt sich in ein Integral verwandeln. Wir führen den Kern

$$(11) \quad \psi_\varrho(x, y; \xi, \eta) = 4 \sum_{m,n=1}^{\infty} g_{mn} \sin(\pi n x) \sin(\pi n y) \sin(\pi m \xi) \sin(\pi n \eta)$$

ein, wobei

$$(12) \quad g_{mn} = g_m \cdot g_n, \quad g_r = \left( \frac{\sin \frac{\pi r \varrho}{2}}{\frac{\pi r \varrho}{2}} \right)^2.$$

Dann ist

$$(13) \quad \tilde{u}(\xi, \eta) = \iint_S \psi_\varrho(\xi, \eta; x, y) u(x, y) dx dy$$

und

$$(14) \quad (P, D^2 \tilde{u})_g = \iint_S \left\{ h^2 \sum_{i,k=1}^N P_{i,k} \psi_{\varrho}(i h, k h; x, y) \right\} (D^2 u) dx dy.$$

Wir werden nun zeigen, daß die Funktion

$$(15) \quad T(x, y) = h^2 \sum_{i,k=1}^N P_{i,k} \psi_{\varrho}(i h, k h; x, y)$$

für  $N \rightarrow \infty$  im Quadratmittel gegen die Funktion  $p(x, y)$  konvergiert. Auch hier gehen wir wieder schrittweise vor. Indem wir

$$(16) \quad \psi_{\varrho}^N(x, y; \xi, \eta) = 4 \sum_{m,n=1}^N g_{mn} \sin(\pi m x) \sin(\pi n y) \sin(\pi m \xi) \sin(\pi n \eta)$$

setzen, definieren wir

$$(17) \quad T_1(x, y) = h^2 \sum_{i,k=1}^N P_{i,k} \psi_{\varrho}^N(i h, k h; x, y).$$

Dann gilt

$$\begin{aligned} T(x, y) - T_1(x, y) \\ = 4 \sum'_{m,n} g_{mn} \sin(\pi m x) \sin(\pi n y) \left\{ h^2 \sum_{i,k=1}^N P_{i,k} \sin(\pi h m i) \sin(\pi h n k) \right\}. \end{aligned}$$

Hier, wie auch im folgenden, bedeutet

$$\sum'_{m,n} = \sum_{m=1}^N \sum_{n=N+1}^{\infty} + \sum_{n=1}^N \sum_{m=N+1}^{\infty} + \sum_{m,n=N+1}^{\infty}.$$

Weiterhin ist

$$\begin{aligned} \|T - T_1\|^2 &= \iint_S (T(x, y) - T_1(x, y))^2 dx dy \\ &= 4 \sum'_{m,n} g_{mn}^2 \left\{ h^2 \sum_{i,k=1}^N P_{i,k} \sin(\pi h m i) \sin(\pi h n k) \right\}^2. \end{aligned}$$

Unter Berücksichtigung der Periodizitätseigenschaften des Ausdrucks in der geschweiften Klammer und der Abschätzung

$$g_m \leq \text{Min} \left( 1, \frac{4}{\pi^2 \varrho^2 m^2} \right)$$

ergibt sich

$$\begin{aligned} & \frac{1}{4} \|T - T_1\|^2 \\ & \leq \sum_{q=1}^{\infty} \sum_{m,n=1}^N g_m^2 g_{n+q(N+1)}^2 \left\{ h^2 \sum_{i,k=1}^N (-1)^{qk} P_{i,k} \sin(\pi h m i) \sin(\pi h n k) \right\}^2 + \\ (18) \quad & + \sum_{p=1}^{\infty} \sum_{m,n=1}^N g_{m+p(N+1)}^2 g_n^2 \left\{ h^2 \sum_{i,k=1}^N (-1)^{pi} P_{i,k} \sin(\pi h m i) \sin(\pi h n k) \right\}^2 + \\ & + \sum_{p,q=1}^{\infty} \sum_{m,n=1}^N g_{m+p(N+1)}^2 g_{n+q(N+1)}^2 \left\{ h^2 \sum_{i,k=1}^N (-1)^{pi+qk} P_{i,k} \sin(\pi h m i) \sin(\pi h n k) \right\}^2 \\ & \leq \frac{1}{4} \left( h^2 \sum_{i,k=1}^N P_{i,k}^2 \right) \left[ \frac{32 h^4}{\pi^4 \varrho^4} \sum_{q=1}^{\infty} \frac{1}{q^4} + \left( \frac{16 h^4}{\pi^4 \varrho^4} \sum_{q=1}^{\infty} \frac{1}{q^4} \right)^2 \right] \end{aligned}$$

und folglich

$$(19) \quad \|T - T_1\| \leq \frac{112}{45} h^2 \varrho^{-2} \|p\|.$$

Nunmehr beachten wir, daß  $P_{i,k}$  mit den Werten der Funktion  $\omega_q^2(p)$  in den Gitterpunkten  $(ih, kh)$  übereinstimmt:

$$(20) \quad P_{i,k} = \sum_{m,n=1}^{\infty} g_{mn} p^{mn} \sin(\pi h m i) \sin(\pi h n k).$$

Dabei steht  $g_{mn}$  für die in (12) erklärten Ausdrücke, und  $p^{mn}$  sind die Fourier-Koeffizienten der Funktion  $p(x, y)$ . Wir schneiden die Reihe (20) ab und definieren die weitere Gitterfunktion

$$(21) \quad P_{i,k}^N = \sum_{m,n=1}^N g_{mn} p^{mn} \sin(\pi h m i) \sin(\pi h n k)$$

und entsprechend eine Funktion

$$(22) \quad T_2(x, y) = h^2 \sum_{i,k=1}^N P_{i,k}^N \psi_q^N(ih, kh; x, y).$$

Die Differenz der Funktionen  $T_1$  und  $T_2$  ist

$$\begin{aligned} T_1(x, y) - T_2(x, y) &= h^2 \sum_{i,k=1}^N \left\{ \sum'_{m,n} g_{mn} p^{mn} \sin(\pi h m i) \sin(\pi h n k) \right\} \times \\ &\quad \times \left\{ 4 \sum_{\mu,v=1}^N g_{\mu v} \sin(\pi h \mu i) \sin(\pi h v k) \sin(\pi \mu x) \sin(\pi v y) \right\} \\ &= \sum_{\mu,v=1}^N g_{\mu v} \sin(\pi \mu x) \sin(\pi v y) \left[ \sum_{\substack{m \equiv \pm \mu \\ n \equiv \pm v \\ (m,n) \neq (\mu,v)}} \pm g_{mn} p^{mn} \right]. \end{aligned}$$

Die Kongruenzen sind modulo  $2(N+1)$  zu nehmen, vgl. (N 28), (N 29).

$$\|T_1 - T_2\|^2 = \frac{1}{4} \sum_{\mu,v=1}^N g_{\mu v}^2 \left[ \sum_{\substack{m \equiv \pm \mu \\ n \equiv \pm v \\ (m,n) \neq (\mu,v)}} \pm g_{mn} p^{mn} \right]^2.$$

Anwenden der Schwarzschen Ungleichung und Berücksichtigung von  $g_{\mu v} \leq 1$  ergeben

$$\|T_1 - T_2\|^2 \leq \frac{1}{4} \left[ \max_{1 \leq \mu, v \leq N} \left( \sum_{\substack{m \equiv \pm \mu \\ n \equiv \pm v \\ (m,n) \neq (\mu,v)}} g_{mn}^2 \right) \right] \sum'_{m,n} (p^{mn})^2.$$

Das Maximum läßt sich ähnlich wie in (18) durch  $\left(\frac{112}{45} h^2 \varrho^{-2}\right)^2$  abschätzen, so daß wir

$$(23) \quad \|T_1 - T_2\| \leq \frac{112}{45} h^2 \varrho^{-2} \|p\|$$

erhalten.

Drücken wir in dem Ausdruck (22) für  $T_2$  die  $P^N$  und  $\psi_\varrho^N$  durch ihre trigonometrischen Darstellungen aus, so erhalten wir

$$T_2(x, y) = \sum_{m, n=1}^N g_{mn}^2 p^{mn} \sin(\pi m x) \sin(\pi n y).$$

Daraus ist ersichtlich, daß  $T_2(x, y)$  nichts anderes als die abgeschnittene Fourier-Entwicklung der Funktion

$$\omega_\varrho^4(p) = \sum_{m, n=1}^{\infty} g_{mn}^2 p^{mn} \sin(\pi m x) \sin(\pi n y)$$

ist. Somit ergibt sich

$$\|T_2 - \omega_\varrho^4(p)\|^2 = \frac{1}{4} \sum_{m, n}' g_{mn}^4 (p^{mn})^2,$$

und da im Summationsgebiet der Summe auf der rechten Seite

$$g_{mn}^4 \leq g_{mn}^2 \leq \left(\frac{16}{\pi^4} h^2 \varrho^{-4}\right)^2$$

ist, weiterhin

$$(24) \quad \|T_2 - \omega_\varrho^4(p)\| \leq \frac{16}{\pi^4} h^2 \varrho^{-4} \|p\|.$$

Als letzte Abschätzung benötigen wir noch die für die Abweichung der Funktion  $\omega_\varrho^4(p)$  von  $p$ . Es ist

$$\|p - \omega_\varrho^4(p)\|^2 = \frac{1}{4} \sum_{m, n=1}^{\infty} (1 - g_{mn}^2)^2 (p^{mn})^2.$$

Unter Verwendung von

$$1 - g_{mn}^2 \leq \text{Min} \left(1, \frac{\pi^2 \varrho^2}{6} (m^2 + n^2)\right)$$

ergibt sich

$$\|p - \omega_\varrho^4(p)\|^2 \leq \frac{\pi^4}{144} \varrho^{4(1-\lambda)} \sum_{m^2 + n^2 \leq \varrho^{-2\lambda}} (p^{mn})^2 + \frac{1}{4} \sum_{m^2 + n^2 > \varrho^{-2\lambda}} (p^{mn})^2$$

( $0 < \lambda < 1$ ). Wir führen die Abkürzung

$$(25) \quad \delta_R(p) = \left[ \frac{1}{4} \sum_{m^2 + n^2 > R^2} (p^{mn})^2 \right]^{\frac{1}{2}}$$

ein. Für jede  $L^2$ -Funktion  $p$  konvergiert  $\delta_R(p)$  nach Null, wenn  $R$  nach Unendlich strebt. Nunmehr wird

$$(26) \quad \|p - \omega_\varrho^4(p)\| \leq \frac{\pi^2}{6} \varrho^{2(1-\lambda)} \|p\| + \delta_{\varrho^{-\lambda}}(p) \quad (0 < \lambda < 1).$$

#### § 4

Wir fassen jetzt die vorangehenden Abschätzungen zusammen. Aus (8), (9), (10) folgt

$$(27) \quad \begin{aligned} & |(P, V^2 U)_g - (P, D^2 \tilde{u})_g| \\ & \leq \gamma_1 \|p\| \|\omega_\varrho^2(f) - F\|_g + \|p\| \|f\| \{\gamma_1 \varepsilon_\varrho(a, b, c) + \gamma_2 h^2 \varrho^{-4}\} \end{aligned}$$

( $\gamma_2 = \gamma_1 + 4C_8$ ). Aus

$$(28) \quad \begin{aligned} & (P, D^2 \tilde{u})_g - \iint_S p(x, y) D^2 u \, dx \, dy \\ & = \iint_S \{(T - T_1) + (T_1 - T_2) + (T_2 - \omega_\varrho^4(p)) + (\omega_\varrho^4(p) - p)\} (D^2 u) \, dx \, dy \end{aligned}$$

finden wir mit Hilfe von (19), (23), (24), (26) und den *a priori* Abschätzungen ( $\mathfrak{N}$  6, 7) für  $\|D^2 u\|$  weiterhin

$$(29) \quad \left| (P, D^2 \tilde{u})_g - \iint_S p(x, y) D^2 u \, dx \, dy \right| \\ \leq \frac{Q}{q^2} \|f\| \left\{ \frac{224}{45} h^2 \varrho^{-2} \|p\| + \frac{16}{\pi^4} h^2 \varrho^{-4} \|p\| + \frac{\pi^2}{6} \varrho \|p\| + \delta_{\varrho^{-\frac{1}{2}}}(p) \right\}.$$

Dabei haben wir  $\lambda = \frac{1}{2}$  gesetzt.

Indem wir die Ungleichungen (27) und (29) kombinieren, erhalten wir schließlich die gewünschte Fehlerabschätzung

$$(30) \quad \left| \iint_S p(x, y) D^2 u \, dx \, dy - h^2 \sum_{i,k=1}^N P_{i,k} \nabla^2 U_{i,k} \right| \\ \leq \gamma_1 \|p\| \|\omega_\varrho^2(f) - F\|_g + \frac{Q}{q^2} \|f\| \delta_{\varrho^{-\frac{1}{2}}}(p) + \\ + \|p\| \|f\| \{ \gamma_1 \varepsilon_\varrho(a, b, c) + \gamma_3 h^2 \varrho^{-4} + \gamma_4 \varrho \} \\ \left( \gamma_3 = \gamma_2 + \frac{224 Q}{45 q^2} + \frac{16 Q}{\pi^4 q^2}, \quad \gamma_4 = \frac{\pi^2 Q}{6 q^2} \right).$$

## § 5

Die Fehlerabschätzung (30) ist unter der Annahme, daß die Funktion  $p(x, y)$  quadratisch integrierbar ist, abgeleitet worden. Dabei hatten wir für die Gitterfunktion  $\{P_{i,k}\}$  die Werte der zweimal geglätteten Funktion  $p$  in den Gitterpunkten genommen. Falls  $p$  eine stetige Funktion ist, dann wird man natürlich  $P_{i,k} = p(ih, kh)$  wählen, so daß die Formel (30) ergänzt werden muß. Wir wollen abschließend die Fehlerabschätzung für diesen Fall angeben.

Die bei der Herleitung der Ungleichung (29) verwendete Abschätzung (26) der Norm von  $\omega_\varrho^4(p) - p$  kann bei stetigem  $p$  durch den Stetigkeitsmodul (vgl.  $\mathfrak{N}$ , Lemma 1) ausgedrückt werden. Zunächst ist klar, daß der Glättungsprozeß den Stetigkeitsmodul nicht verschlechtert:

$$\varepsilon_\tau(\omega_\varrho^l(p)) \leq \varepsilon_\tau(p) \quad (l = 1, 2, \dots).$$

Weiterhin ergibt sich

$$\max_{(x,y) \in \bar{S}} |\omega_\varrho^{l+2}(p) - \omega_\varrho^l(p)| \leq \varepsilon_\varrho(\omega_\varrho^l(p)) \leq \varepsilon_\varrho(p).$$

Folglich haben wir

$$\|\omega_\varrho^4(p) - p\| \leq \max_{(x,y) \in \bar{S}} |\omega_\varrho^4(p) - p| \leq 2\varepsilon_\varrho(p).$$

Lemma 7 von  $\mathfrak{N}$  liefert (im Falle  $v \equiv 1$ )

$$\|\omega_\varrho^2(p) - p\|_g \leq 4\varepsilon_\varrho(p).$$

Somit wird unter Benutzung des Hilfssatzes von § 4

$$(31) \quad |(P, \nabla^2 U)_g - (p, \nabla^2 U)_g| = |(\omega_\varrho^2(p) - p, \nabla^2 U)_g| \\ \leq 4\varepsilon_\varrho(p) \|\nabla^2 U\|_g \leq \frac{4Q}{q^2} \varepsilon_\varrho(p) \|F\|_g \\ \leq \frac{16Q}{q^2} \varepsilon_\varrho(p) \|f\| + \frac{4Q}{q^2} \varepsilon_\varrho(p) \|F - \omega_\varrho^2(f)\|_g.$$

Kombination der Ungleichungen (30) und (31) gibt

$$\begin{aligned}
 (32) \quad & \left| \iint_S p(x, y) D^2 u \, dx \, dy - h^2 \sum_{i, k=1}^N p(i h, k h) V^2 U_{i, k} \right| \\
 & \leq \left( \gamma_1 \|p\| + \frac{4Q}{q^2} \varepsilon_q(p) \right) \|F - \omega_e^2(f)\|_g + \frac{18Q}{q^2} \|f\| \varepsilon_q(p) + \\
 & \quad + \|p\| \|f\| \{ \gamma_1 \varepsilon_q(a, b, c) + \gamma_3 h^2 q^{-4} \}.
 \end{aligned}$$

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# Some Investigations of a Linearized Theory for Unsteady Cavity Flows

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## Contents

	Page
1. Introduction . . . . .	315
Part I. Reentrant-jet flow with unsteady perturbations	
2. Formulation of the problem . . . . .	316
3. Steady flow . . . . .	317
4. Unsteady motion as a perturbation of steady flow . . . . .	321
5. Additional conditions . . . . .	327
6. First order approximation for small breadth . . . . .	330
Part II. Linearized theory of unsteady cavity flow	
7. Summary . . . . .	334
8. Linearized theory of steady flow . . . . .	335
9. Unsteady flow; method of the complex velocity . . . . .	337
10. Linearized unsteady cavity flow as a first order approximation . . . . .	340
11. Unsteady flow; method of the complex acceleration-potential . . . . .	341
12. The drag . . . . .	344
13. Conclusions . . . . .	345
References . . . . .	346

## 1. Introduction

The literature concerning unsteady motion of a body in a fluid with free streamlines can be divided roughly into three categories.

The first category deals with exact solutions. In that case it is necessary to restrict the field to unsteady motions of a special type. VON KÁRMÁN [1] was the first to take that direction. Later publications appeared by GILBARG [2] and WOODS [3].

In the second category we shall class those articles in which the unsteady motion is considered as a perturbation of a steady flow. In this way WOODS [4] investigated the unsteady motion of a flat plate according to the Helmholtz-Kirchhoff model. In [5] WU gave a formulation of a boundary-value problem for the Roshko model (also named transition model). Closely connected with this is the research on the stability of free streamline flows. ABLOW & HAYES [6] published a report on the subject. Their results were extended by FOX & MORGAN in [7].

The third category contains papers in which the whole flow is linearized. TULIN [8] was the first to apply the method of linearization; he considered a steady flow. PARKIN [9] solved the unsteady problem of an infinite cavity

behind an inclined flat plate and gave a formulation of the finite cavity case. In [10] and [5] Wu developed a theory for the unsteady finite cavity case. TIMMAN [11] and the author [12] treated the case of a partially cavitated inclined flat plate. The unsteady motion was considered small with respect to the steady flow (second linearization). In order to get a closed cavity it was necessary to admit a discontinuity in the unsteady vertical perturbation velocity in the wake of the body. Another possibility<sup>1</sup> is to drop the condition that the cavity be closed in the unsteady case and to require that the normal component of the unsteady perturbation velocity be continuous in the wake. The last possibility will be investigated in this paper by comparing the linearized cavity flow with the corresponding reentrant-jet flow. For a steady cavity flow it has been proved that the linearized theory yields a first order approximation to reentrant-jet flow theory [13]. From the following it will appear that also in the unsteady case a linearized theory can be formulated in such a way as to become a first-order approximation to the reentrant-jet theory. We shall prove this connection, just as in [13], for the simplest case, that of a flat plate moving perpendicularly to its plane.

The paper is divided into two parts. In the first part we consider a non-linear steady reentrant-jet flow. The unsteady motion is treated by first-order perturbation theory. Since in this theory an arbitrary motion can be considered as a superposition of harmonic oscillations, only a harmonic time-dependence is introduced.

In the second part we treat the linearized problem. The solution is obtained by two different methods. The first uses the complex velocity and is directly connected with the method of the first part. The second works with the complex acceleration-potential and offers the best prospect for more complicated problems.

Finally, it should be remarked that in this introduction we have omitted papers that deal with the initial stage of impulsive unsteady motion. Several Russian authors have worked in that direction, *e.g.*, GUREVICH [14]. Their investigations are directed towards the determination of induced mass of the cavity (see BIRKHOFF & ZARANTONELLO [15]). In this paper, however, we are interested in the unsteady motion as a whole.

## Part I. Reentrant-jet flow with unsteady perturbations

### *2. Formulation of the problem*

A flat plate is placed in a flow of an incompressible non-viscous fluid with uniform velocity perpendicular to the plate. Behind the plate a cavity with reentrant jet has formed. The jet is supposed to disappear on a second sheet of the physical plane considered as a Riemann surface. This artifice is used to represent within the framework of potential theory the dissipation of energy that is observed at the rear end and in the interior of the cavity (see BIRKHOFF & ZARANTONELLO [15]). Here we prefer the reentrant-jet model to the Riabouchinsky model, since the former shows more resemblance to experimental observations (see EISENBERG [16]). When the plate performs an unsteady

<sup>1</sup> This possibility emerged from a discussion with Dr. P. EISENBERG and Dr. M. P. TULIN.

motion, the shape and size of the cavity will change. If we assume that the deviation from the steady position is small in a certain sense, this unsteady motion can be considered as a perturbation of the original steady flow (linearization of the unsteady effects).

First we treat the stationary flow, and then we investigate the effect of the unsteady motion.

### 3. Steady flow

The steady symmetric flow about a perpendicular flat plate with reentrant jet is shown in Figure 1.

The Cartesian coordinate system has been chosen in such a way that the middle of the plate lies at the origin, and the velocity of the fluid at infinity has the direction of the positive  $x$ -axis. We denote half the breadth of the plate by  $b$ . The fluid is assumed to be incompressible and non-viscous, and the motion, steady and irrotational. The relation between the pressure and velocity is then given by the Bernoullian law:

$$p_{\infty} + \frac{1}{2} \rho U^2 = p + \frac{1}{2} \rho q^2 = p_c + \frac{1}{2} \rho q_c^2, \quad (3.1)$$

where  $q$  = magnitude of velocity,

$p$  = pressure,

$U$  = velocity at infinity,

$p_{\infty}$  = pressure at infinity,

$q_c$  = velocity on the cavity,

$p_c$  = constant pressure on the cavity,

$\rho$  = constant density of the fluid.

We choose the units of length and time in such a way that the constant fluid velocity  $q_c$  on the cavity is equal to 1. The susceptibility of the fluid for the occurrence of cavitation is characterized by the cavitation number  $\sigma$ :

$$\sigma = \frac{p_{\infty} - p_c}{\frac{1}{2} \rho U^2}. \quad (3.2)$$

From (3.1) and (3.2) it follows that

$$U = (1 + \sigma)^{-\frac{1}{2}}. \quad (3.3)$$

The  $x$  and  $y$ -components  $u$  and  $v$  of the velocity vector  $q$  satisfy the conditions of incompressibility and irrotationality:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0. \end{aligned} \quad (3.4)$$

These equations show that  $w = u - iv$  is a holomorphic function of  $z = x + iy$  in the interior of the fluid. On account of (3.4) we can define a velocity potential  $\varphi$  and a stream function  $\psi$  such that

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \\ v &= \frac{\partial \varphi}{\partial y} = - \frac{\partial \psi}{\partial x}. \end{aligned} \quad (3.5)$$

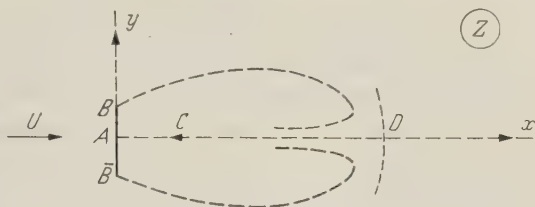


Fig. 1. Reentrant-jet flow about a flat plate

Then  $\Phi = \varphi + i\psi$  is a holomorphic function of  $z = x + iy$ , and

$$w = \frac{d\Phi}{dz}. \quad (3.6)$$

For the moment we consider only the upper half of the physical plane, *i.e.*, the upper half of the region occupied by the fluid. As a result of the presence of the reentrant jet this region is two-sheeted. Since the region is simply connected,

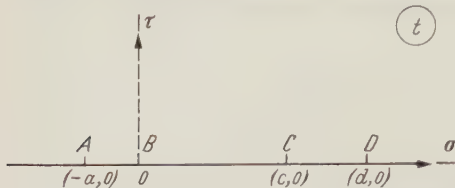


Fig. 2. Conformal image on the upper half of the  $t$ -plane

we can map it conformally onto the upper half of a  $t$ -plane ( $t = \sigma + i\tau$ )<sup>2</sup>.

The images of three points on the boundary can be fixed arbitrarily. We choose the image of  $z = \infty$  as  $t = \infty$ , the image of  $B$  as  $t = 0$  and the image of  $A$  as  $t = -a$ , where  $a$  is a fixed positive number (see Figure 2).

In paragraphs a) and b) below we determine  $\Phi$  and  $w$  as a function of  $t$ . The conformal mapping giving  $z$  as a function of  $t$  is then given by

$$dz = w^{-1} d\Phi = w^{-1} \frac{d\Phi}{dt} dt,$$

or

$$z = \int_{-a}^t w^{-1} \frac{d\Phi}{dt} dt. \quad (3.7)$$

a) *Determination of  $\Phi$  as a function of  $t$ .* We consider  $\frac{d\Phi}{dt}$ .  $\frac{d\Phi}{dt}$  is a holomorphic function of  $t$  in the upper half-plane. Since the real axis is the image of streamlines (where  $\psi = \text{constant}$ ),  $\frac{d\Phi}{dt}$  is real along the real axis. By means of the reflection principle of SCHWARZ we can continue  $\frac{d\Phi}{dt}$  analytically into the lower half-plane.  $\frac{d\Phi}{dt}$  can have singularities on the real axis. These cannot be essential singularities (for a proof see BIRKHOFF & ZARANTONELLO [15]). Singularities can occur only at  $t = \infty$  and  $t = c$ . From a consideration of the local behavior it follows that  $\frac{d\Phi}{dt}$  remains finite at  $t = \infty$  and behaves like  $\frac{C}{t-c}$  at  $t = c$ . As a result of the branching of the streamlines at the point  $D$ ,  $\frac{d\Phi}{dt}$  will have a simple zero at  $t = d$ . Thus

$$\frac{d\Phi}{dt} = K \frac{t-d}{t-c}, \quad (3.8)$$

where  $K$  is a real constant; hence

$$\Phi = K[t - (d-c) \log(t-c)] + \text{constant}. \quad (3.9)$$

b) *Determination of  $w$  as a function of  $t$ .* We introduce  $\Omega = \log w = \log q - i\vartheta$ , where  $\vartheta$  is the angle between the velocity vector and the positive  $x$ -axis.  $\Omega$  is

<sup>2</sup> Of course this  $\sigma$  is different from the  $\sigma$  defined by (3.2). This will cause no confusion.

a holomorphic function of  $t$  in the upper half-plane, where it satisfies the following mixed boundary conditions:

$$\begin{aligned}
 -\infty < \sigma < -a, & \quad \vartheta = 0 & \text{or} & \quad \text{Im } \Omega = 0, \\
 -a < \sigma < 0, & \quad \vartheta = \frac{1}{2}\pi & \text{or} & \quad \text{Im } \Omega = -\frac{1}{2}\pi, \\
 0 < \sigma < c, & \quad q = 1 & \text{or} & \quad \text{Re } \Omega = 0, \\
 c < \sigma < d, & \quad \vartheta = -\pi & \text{or} & \quad \text{Im } \Omega = \pi, \\
 d < \sigma < \infty, & \quad \vartheta = 0 & \text{or} & \quad \text{Im } \Omega = 0.
 \end{aligned} \tag{3.10}$$

For the solution of this boundary-value problem we continue  $\Omega$  analytically into the lower half-plane by means of the reflection principle of SCHWARZ:

$$\Omega(\bar{t}) = \overline{\Omega(t)}. \tag{3.11}$$

The boundary-value problem is now equivalent to the following Hilbert problem (see MUSKHELISHVILI [17]), viz, to determine  $\Omega(t)$ , holomorphic in the upper and lower half-planes, satisfying the following linear relations on the real axis:

$$\begin{aligned}
 -\infty < \sigma < -a, & \quad \Omega^+ - \Omega^- = 0, \\
 -a < \sigma < 0, & \quad \Omega^+ - \Omega^- = -\pi i, \\
 0 < \sigma < c, & \quad \Omega^+ + \Omega^- = 0, \\
 c < \sigma < d, & \quad \Omega^+ - \Omega^- = 2\pi i, \\
 d < \sigma < \infty, & \quad \Omega^+ - \Omega^- = 0.
 \end{aligned} \tag{3.12}$$

Here  $\Omega^+$  and  $\Omega^-$  denote the boundary values that are assumed by  $\Omega$  in approaching the real axis from the upper and lower sides. Moreover,  $\Omega$  has to be continuous at the points  $t=0$  and  $t=c$ . At infinity  $\Omega$  must remain finite.

We consider first the homogeneous Hilbert problem, i.e., the above-mentioned problem with the right-hand sides in (3.12) replaced by 0. A solution  $\Omega_h$  of this homogeneous problem is

$$\Omega_h = \sqrt{t(t-c)}. \tag{3.13}$$

The cut imposed so as to make the square root single-valued is taken along the real axis from  $t=0$  to  $t=c$ . The branch is chosen in such a way that

$$\sqrt{t(t-c)} \sim t \quad \text{as } t \rightarrow \infty.$$

The Hilbert problem (3.12) can now be formulated as follows: Determine  $\Omega(t)$  in such a way as to satisfy on the real axis the relations

$$\begin{aligned}
 -\infty < \sigma < -a, & \quad \left[ \frac{\Omega}{\Omega_h} \right]^+ - \left[ \frac{\Omega}{\Omega_h} \right]^- = 0, \\
 -a < \sigma < 0, & \quad \left[ \frac{\Omega}{\Omega_h} \right]^+ - \left[ \frac{\Omega}{\Omega_h} \right]^- = \frac{\pi i}{\sqrt{-\sigma(c-\sigma)}}, \\
 0 < \sigma < c, & \quad \left[ \frac{\Omega}{\Omega_h} \right]^+ - \left[ \frac{\Omega}{\Omega_h} \right]^- = 0, \\
 c < \sigma < d, & \quad \left[ \frac{\Omega}{\Omega_h} \right]^+ - \left[ \frac{\Omega}{\Omega_h} \right]^- = \frac{2\pi i}{\sqrt{\sigma(\sigma-c)}}, \\
 d < \sigma < \infty, & \quad \left[ \frac{\Omega}{\Omega_h} \right]^+ - \left[ \frac{\Omega}{\Omega_h} \right]^- = 0.
 \end{aligned} \tag{3.14}$$

By means of the formulas of PLEMELJ (see MUSKHELISHVILI [17]) we find that

$$\Omega = \sqrt{t(t-c)} \frac{1}{2\pi i} \left[ \int_{-a}^0 \frac{\pi i}{\sqrt{-\sigma(c-\sigma)}(\sigma-t)} d\sigma + \int_c^d \frac{2\pi i d\sigma}{\sqrt{\sigma(\sigma-c)}(\sigma-t)} \right]. \quad (3.15)$$

After some calculation we arrive at

$$\Omega = \frac{1}{2} \log \frac{[\sqrt{t(c+a)} - \sqrt{a(t-c)}]}{[\sqrt{t(c+a)} + \sqrt{a(t-c)}]} + \log \frac{[\sqrt{d(t-c)} - \sqrt{t(d-c)}]}{[\sqrt{d(t-c)} + \sqrt{t(d-c)}]}, \quad (3.16)$$

and

$$w = \frac{[\sqrt{t(c+a)} - \sqrt{a(t-c)}]^{\frac{1}{2}}}{[\sqrt{t(c+a)} + \sqrt{a(t-c)}]^{\frac{1}{2}}} \cdot \frac{[\sqrt{d(t-c)} - \sqrt{t(d-c)}]}{[\sqrt{d(t-c)} + \sqrt{t(d-c)}]}. \quad (3.17)$$

The expression for  $z$  as a function of  $t$  can now be written as

$$z = K \int_{-a}^t \frac{[\sqrt{t(c+a)} + \sqrt{a(t-c)}]^{\frac{1}{2}}}{[\sqrt{t(c+a)} - \sqrt{a(t-c)}]^{\frac{1}{2}}} \frac{[\sqrt{d(t-c)} + \sqrt{t(d-c)}]}{[\sqrt{d(t-c)} - \sqrt{t(d-c)}]} \frac{t-d}{t-c} dt. \quad (3.18)$$

*Remark.* (3.16) and (3.17) represent  $\Omega$  and  $w$  also in the lower half of the physical plane when we map it conformally onto the lower half of the  $t$ -plane by means of (3.18).

In the solution, which is the combination of (3.17) and (3.18), three unknown constants, *viz.*  $K$ ,  $c$  and  $d$  still occur. These unknowns are determined by the following conditions:

1. at  $t = \infty$  we must have  $w = U = (1 + \sigma)^{-\frac{1}{2}}$ ,
2.  $z$  has to be a single-valued function of  $t$ ,
3. the distance between  $A$  and  $B = b =$  half the breadth of the flat plate.

We first formulate Conditions 2 and 3 mathematically. The condition that  $z$  should be a single-valued function of  $t$  reads

$$\oint_C e^{-\Omega} \frac{d\Phi}{dt} dt = 0, \quad (3.19)$$

where  $C$  is a closed contour surrounding the plate and cavity, *i.e.*, surrounding the part of the real axis that lies between  $-a$  and  $d$ . Application of the theorem of residues to the exterior of  $C$  gives

$$\left\{ \text{res } e^{-\Omega} \frac{d\Phi}{dt} \right\}_{t=\infty} = 0. \quad (3.20)$$

Condition 3 can be written as

$$\int_{-a}^0 e^{-\Omega} \frac{d\Phi}{dt} dt = b i. \quad (3.21)$$

We now apply the three conditions to the solution obtained. Condition 1 gives us

$$(1 + \sigma)^{-\frac{1}{2}} = \frac{[\sqrt{c+a} - \sqrt{a}]^{\frac{1}{2}}}{[\sqrt{c+a} + \sqrt{a}]^{\frac{1}{2}}} \cdot \frac{[\sqrt{d} - \sqrt{d-c}]}{[\sqrt{d} + \sqrt{d-c}]}. \quad (3.22)$$

Condition 2 furnishes the equation

$$-\frac{1}{2} \sqrt{a} \sqrt{c+a} - \frac{c\sqrt{d}}{\sqrt{d}+\sqrt{d-c}} + c = 0. \quad (3.23)$$

From Condition 3 we obtain

$$K \int_{-a}^0 \frac{[\sqrt{\sigma(c+a)} + \sqrt{a(\sigma-c)}]^{\frac{1}{2}}}{[\sqrt{\sigma(c+a)} - \sqrt{a(\sigma-c)}]^{\frac{1}{2}}} \cdot \frac{[\sqrt{d(\sigma-c)} + \sqrt{\sigma(d-c)}]}{[\sqrt{d(\sigma-c)} - \sqrt{\sigma(d-c)}]} \frac{\sigma-d}{\sigma-c} d\sigma = bi. \quad (3.24)$$

Elimination of  $\sqrt{\frac{d-c}{d}}$  between (3.22) and (3.23) gives the following relation between  $c$  and  $\sigma$ :

$$1 + \sigma = \left[ \sqrt{\frac{a}{c}} + \sqrt{1 + \frac{a}{c}} \right]^2 \left[ 1 - \sqrt{\frac{a}{c}} \sqrt{1 + \frac{a}{c}} \right]^{-2}. \quad (3.25)$$

Moreover it can be shown that

$$\sqrt{\frac{d-c}{d}} = \frac{1}{2} \sqrt{\frac{a}{c}} \sqrt{1 + \frac{a}{c}} \left[ 1 - \frac{1}{2} \sqrt{\frac{a}{c}} \sqrt{1 + \frac{a}{c}} \right]^{-1}, \quad (3.26)$$

and

$$d - c = \frac{c}{4} \frac{a}{c} \left( 1 + \frac{a}{c} \right) \left[ 1 - \sqrt{\frac{a}{c}} \sqrt{1 + \frac{a}{c}} \right]^{-1}, \quad (3.27)$$

The integral in (3.24) can be expressed in terms of elementary functions, but since the result is rather unwieldy, it is omitted. (3.25) gives the relation between  $c$  and  $\sigma$ . From (3.27) follows the connection between  $d$  and  $\sigma$ . Finally (3.24) determines  $K$  as a function of  $b$  and  $\sigma$ . Thus the solution for the steady case is completely determined.

#### 4. Unsteady motion as a perturbation of steady flow

Let the flat plate perform an oscillation of the form  $h e^{j\omega T}$  normal to the main flow. By  $T$  we denote the time variable;  $h$  is the amplitude of the oscillation. The deviation is taken positive in the direction of the negative  $x$ -axis. Thus the normal velocity in the direction of the negative  $x$ -axis is equal to  $j\omega h e^{j\omega T}$ .

*Remark.* It is tacitly assumed that in all expressions containing exponential terms complex with respect to  $j$ , we must take the real part in order to get the true time behavior. In the formulas this real part is denoted by "re". Since we linearize the unsteady part of the motion, this procedure will cause no trouble. Of course the imaginary units  $i$  and  $j$  are unconnected; we have  $i^2 = -1$  and  $j^2 = -1$ , but  $ij$  is not reducible. The real part with respect to  $i$  is denoted by "Re", the imaginary part is denoted by "Im".

We now suppose that  $\omega h$  is small with respect to  $q_c$  and that  $h$  is small with respect to  $b$ . In this case the unsteady motion may be considered as a perturbation of the steady flow. In order to formulate these conditions more rigorously we introduce two non-dimensional parameters  $\varepsilon_1$  and  $\varepsilon_2$ :

$$\varepsilon_1 = \frac{\omega h}{q_c} \text{ (velocity parameter),} \quad (4.1)$$

and

$$\varepsilon_2 = \frac{h}{b} \text{ (amplitude parameter).} \quad (4.2)$$

The conditions mentioned above can be expressed by

$$\varepsilon_1 \ll 1 \text{ and } \varepsilon_2 \ll 1. \quad (4.3)$$

To express the relative magnitudes of  $\varepsilon_1$  and  $\varepsilon_2$  we introduce the parameter

$$\varepsilon_3 = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\omega b}{q_c} \quad (\text{frequency parameter})$$

and require that

$$\varepsilon_3 \gg 1. \quad (4.4)$$

The reason for this requirement will be discussed below under a). We shall neglect, as usual, second and higher powers in the disturbances in comparison with first order terms, and moreover terms proportional to  $\varepsilon_2$  in comparison with those proportional to  $\varepsilon_1$ . This last procedure is motivated by (4.4).

The boundary conditions will now be formulated.

a) On the flat plate the boundary condition is (see Figure 3)

$$\vartheta - \frac{1}{2}\pi = \arcsin \frac{j\omega h e^{j\omega T}}{q}. \quad (4.5)$$

We put  $\varphi = \varphi_0 + \varphi_1$ ,  $\psi = \psi_0 + \psi_1$ ,  $q = q_0 + q_1$ ,  $\vartheta = \vartheta_0 + \vartheta_1$  and  $p = p_0 + p_1$ , where quantities referring to the steady flow are denoted by the suffix 0, and those referring to the first order terms of the unsteady perturbation by the suffix 1. These first order terms include, for the time being, the terms which are linear with respect to  $\varepsilon_1$  or  $\varepsilon_2$ .

In a first order approximation (4.5) may be replaced by

$$\vartheta_1 = \frac{\pi}{2} - \vartheta_0 + \frac{j\omega h e^{j\omega T}}{q_0}. \quad (4.6)$$

This approximation is not valid near the stagnation point, but nevertheless, as is customary in linearized theories, we assume it to be valid there. We wish to apply this condition to the unperturbed position of the flat plate. In a first order approximation it is true that on the flat plate

$$\vartheta_0 = \frac{\pi}{2} - \frac{\partial \vartheta_0}{\partial x} h, \quad (4.7)$$

where the derivative  $\frac{\partial \vartheta_0}{\partial x}$  must be taken at the unperturbed position of the plate. Using the fact that  $\log q_0 - i\vartheta_0$  is a holomorphic function of  $z$ , we may reduce (4.7) as follows:

$$\begin{aligned} \vartheta_0 &= \frac{\pi}{2} - \frac{\partial \log q_0}{\partial y} h = \frac{\pi}{2} - \frac{\partial \log q_0}{\partial \varphi_0} q_0 h \\ &= \frac{\pi}{2} - \frac{\partial q_0}{\partial \varphi_0} h = \frac{\pi}{2} - \frac{\partial q_0}{\partial \sigma} \frac{1}{K} \frac{\sigma - c_0}{\sigma - d_0} h. \end{aligned} \quad (4.8)$$

In this reduction use has been made of (3.8).

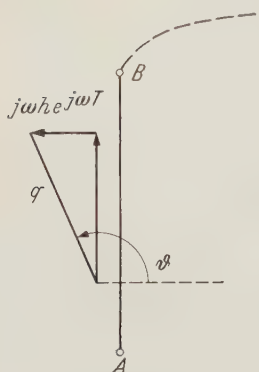


Fig. 3. Unsteady velocity on the flat plate

*Remark.*  $c_0$  and  $d_0$  denote the steady values of  $c$  and  $d$ . This distinction is necessary, since as a result of the unsteady motion  $c$  and  $d$  will change with time.

Boundary condition (4.6) may now be written as

$$\vartheta_1 = \frac{j\omega h e^{j\omega T}}{q_0} + \frac{\partial q_0}{\partial \sigma} \frac{1}{K} \frac{\sigma - c_0}{\sigma - d_0} h. \quad (4.9)$$

In accordance with condition (4.4) the second term in (4.9) may be neglected in comparison to the first term at the right-hand side. Thus the final form of the boundary condition on the flat plate is

$$\vartheta_1 = \frac{j\omega h e^{j\omega T}}{q_0}. \quad (4.10)$$

The condition must be applied at the unperturbed position of the plate, *i.e.*, on the interval  $-a < \sigma < 0$  of the real axis of the  $t$ -plane;  $q_0$  as a function of  $\sigma$  is given by

$$q_0 = \frac{[Va(c_0 - \sigma) - V - \sigma(c_0 + a)]^{\frac{1}{2}} [Vd_0(c_0 - \sigma) - V - \sigma(d_0 - c_0)]}{[Va(c_0 - \sigma) + V - \sigma(c_0 + a)]^{\frac{1}{2}} [Vd_0(c_0 - \sigma) + V - \sigma(d_0 - c_0)]}. \quad (4.11)$$

This follows from (3.17). In the sequel we shall write  $Q(\sigma)$  for  $\frac{1}{q_0(\sigma)}$ .

We shall now discuss briefly the difficulties encountered in the case when  $\varepsilon_1$  and  $\varepsilon_2$  are of the same order of magnitude. Then the second term on the right-hand side of (4.9) must be retained. Since in the neighborhood of  $\sigma = 0$  the expansion of  $q_0$ , as derived from (4.11), begins as

$$q_0 = 1 - \left[ \sqrt{\frac{c_0 + a}{a c_0}} + 2 \sqrt{\frac{d_0 - c_0}{d_0 c_0}} \right] \sqrt{-\sigma} + \dots, \quad (4.12)$$

the behavior of  $\frac{\partial q_0}{\partial \sigma}$  there is seen from the following term:

$$\frac{\partial q_0}{\partial \sigma} = \left[ \frac{1}{2} \sqrt{\frac{c_0 + a}{a c_0}} + \sqrt{\frac{d_0 - c_0}{d_0 c_0}} \right] \frac{1}{\sqrt{-\sigma}} + \dots. \quad (4.13)$$

Thus  $\frac{\partial q_0}{\partial \sigma}$  becomes infinite at the edge of the flat plate. This fact can be formulated physically by the statement that there the acceleration of the fluid is infinite. The occurrence of this singularity is known in the potential theory of steady cavities separating from sharp edges of obstacles. This singular behavior, however, is not realistic for actual fluids because of the viscosity. Now solving the boundary-value problem by taking (4.9) instead of (4.10) as the boundary condition on the flat plate, we can see from a formula similar to (4.40) at the end of this section that the perturbation pressure  $p_1$  is no longer continuous at the edge of the plate. From physical reasons, however, we should expect continuity there. Therefore it is plausible to conjecture that in unsteady motion when  $\varepsilon_1$  and  $\varepsilon_2$  are of the same order of magnitude, the point where the cavity separates from the plate is no longer fixed at the sharp edge of the plate. Since we wish to restrict our attention to cavity flows with fixed separation points, we have introduced the assumption (4.4).

b) On the boundary of the cavity we consider the Bernoullian law for unsteady flow:

$$\frac{\partial \varphi}{\partial T} + \frac{1}{2} q^2 + \frac{p}{\varrho} = C(T). \quad (4.14)$$

Splitting up all quantities into steady and unsteady parts, we may replace (4.14) to a first order of approximation by

$$\frac{\partial \varphi_1}{\partial T} + \frac{1}{2} q_0^2 + q_0^2 \frac{q_1}{q_0} + \frac{p_0}{\varrho} + \frac{p_1}{\varrho} = C(T). \quad (4.15)$$

Since  $\frac{1}{2} q_0^2 + \frac{p_0}{\varrho} = \text{constant}$  throughout the entire region occupied by the fluid, we may reduce (4.15) to

$$\frac{\partial \varphi_1}{\partial T} + q_0^2 \frac{q_1}{q_0} + \frac{p_1}{\varrho} = D(T). \quad (4.16)$$

If  $\frac{q_1}{q_0} = r_1$ , the Bernoullian law reads

$$\frac{\partial \varphi_1}{\partial T} + q_0^2 r_1 + \frac{p_1}{\varrho} = D(T). \quad (4.17)$$

*Remark.* In partial differentiation with respect to  $T$  the space variables are kept constant. This is equivalent to keeping  $\varphi_0$  and  $\psi_0$  constant.

On the boundary of the cavity we have  $p_1 = 0$ . Thus

$$\frac{\partial \varphi_1}{\partial T} + q_0^2 r_1 = D(T). \quad (4.18)$$

We wish to apply this equation on the boundary of the unperturbed cavity. Letting points correspond to each other by orthogonal projection, we find that (4.18) may be replaced to a first order of approximation by

$$\frac{\partial \varphi_1}{\partial T} + r_1 = D(T), \quad \text{valid on the unperturbed cavity, where } q_0 = 1. \quad (4.19)$$

On the boundary of the unperturbed cavity, however,

$$\psi_0 = \text{const.}$$

Differentiating (4.19) with respect to  $\varphi_0$ , we get

$$\frac{\partial}{\partial T} \frac{\partial \varphi_1}{\partial \varphi_0} + \frac{\partial r_1}{\partial \varphi_0} = 0. \quad (4.20)$$

We prove first that

1.  $r_1 - i\vartheta_1$  is a holomorphic function of  $t$ , and

$$\frac{\partial \varphi_1}{\partial \varphi_0} = r_1. \quad (4.21)$$

2. Putting  $\Phi = \Phi_0 + \Phi_1$ , we have

$$\frac{d\Phi_1}{d\Phi_0} = \frac{d\Phi}{d\Phi_0} - 1 = \frac{d\Phi}{dz} \frac{dz}{d\Phi_0} - 1 = \frac{w}{w_0} - 1. \quad (4.22)$$

To a first order of approximation

$$w = e^{\Omega} = e^{\Omega_0 + \Omega_1} = e^{\Omega_0} (1 + \Omega_1) = w_0 (1 + \Omega_1),$$

or

$$\frac{w}{w_0} - 1 = \Omega_1. \quad (4.23)$$

Combination of (4.22) and (4.23) gives

$$\Omega_1 = \frac{d\Phi_1}{d\Phi_0}. \quad (4.24)$$

Always to the same order of approximation,

$$\begin{aligned} \Omega &= \log w = \log q - i\vartheta = \log \left\{ q_0 \left( 1 + \frac{q_1}{q_0} \right) \right\} - i\vartheta_0 - i\vartheta_1 \\ &= \log q_0 + r_1 - i\vartheta_0 - i\vartheta_1 = \Omega_0 + r_1 - i\vartheta_1. \end{aligned} \quad (4.25)$$

Since  $\Omega$  and  $\Omega_0$  are holomorphic functions of  $t$ , we conclude that

$$\Omega_1 = r_1 - i\vartheta_1 \quad (4.26)$$

is a holomorphic function of  $t$ . From (4.24) and (4.26) follows finally that

$$\frac{\partial \varphi_1}{\partial \varphi_0} = \operatorname{Re} \Omega_1 = r_1. \quad \text{q.e.d.} \quad (4.27)$$

On the unperturbed cavity the boundary condition is now

$$\frac{\partial r_1}{\partial T} + \frac{\partial r_1}{\partial \varphi_0} = 0, \quad (4.28)$$

or

$$r_1 = G(T - \varphi_0), \quad (4.29)$$

i.e.,  $r_1$  is a function of  $T - \varphi_0$ .

In the case of harmonic time-dependence we get

$$r_1 = g e^{j\omega(T - \varphi_0)}, \quad (4.30)$$

where  $g$  is a still unknown constant, real with respect to  $i$ , but possibly complex with respect to  $j$ .

In considering here this harmonic time-dependence, we anticipate, since justification follows only by (5.15), expressing  $g$  as a linear function of  $\omega h$ .

For  $\Omega = \Omega_0 + \Omega_1$  the following mixed boundary-value problem can now be formulated: Determine  $\Omega(t)$ , holomorphic in the upper half-plane, in such a way that on the real axis it assumes the following boundary values:

$$\begin{aligned} -\infty < \sigma < -a, & \quad \vartheta = 0 & \quad \text{or} \quad \operatorname{Im} \Omega = 0, \\ -a < \sigma < 0, & \quad \vartheta = \frac{1}{2}\pi + j\omega h Q(\sigma) e^{j\omega T} & \quad \text{or} \quad \operatorname{Im} \Omega = \frac{1}{2}\pi - j\omega h Q(\sigma) e^{j\omega T}, \\ 0 < \sigma < c, & \quad \log q_0 + r_1 = g e^{j\omega(T - \varphi_0)} & \quad \text{or} \quad \operatorname{Re} \Omega = g e^{j\omega(T - \varphi_0)}, \\ c < \sigma < d, & \quad \vartheta = -\pi & \quad \text{or} \quad \operatorname{Im} \Omega = \pi, \\ d < \sigma < \infty, & \quad \vartheta = 0 & \quad \text{or} \quad \operatorname{Im} \Omega = 0. \end{aligned} \quad (4.31)$$

Putting  $\Omega(\bar{t}) = \overline{\Omega(t)}$  according to the Schwarzian reflection principle, we can transform the mixed boundary-value problem into a Hilbert problem with the following linear relations along the real axis:

$$\begin{aligned} -\infty < \sigma < -a, & \quad \Omega^+ - \Omega^- = 0, \\ -a < \sigma < 0, & \quad \Omega^+ - \Omega^- = -\pi i - 2j\omega h Q(\sigma) e^{j\omega T}, \\ 0 < \sigma < c, & \quad \Omega^+ + \Omega^- = 2g e^{j\omega(T - \varphi_0)}, \\ c < \sigma < d, & \quad \Omega^+ - \Omega^- = 2\pi i, \\ d < \sigma < \infty, & \quad \Omega^+ - \Omega^- = 0. \end{aligned} \quad (4.32)$$

*Remark.* To a first order of approximation  $g$  can be taken equal to zero at  $c_0 < \sigma < c$  if  $c_0 < c$ . Putting  $c = c_0 + c_1$  and  $d = d_0 + d_1$ , we can split this Hilbert problem (4.32) into the following Hilbert problems for  $\Omega_p$  and  $\Omega_s$ , where

$$\Omega = \Omega_p + \Omega_s:$$

$$\begin{aligned} -\infty < \sigma < -a, & \quad \Omega_p^+ - \Omega_p^- = 0, \\ -a < \sigma < 0, & \quad \Omega_p^+ - \Omega_p^- = -\pi i, \\ 0 < \sigma < c, & \quad \Omega_p^+ + \Omega_p^- = 0, \\ c < \sigma < d, & \quad \Omega_p^+ - \Omega_p^- = 2\pi i, \\ d < \sigma < \infty, & \quad \Omega_p^+ - \Omega_p^- = 0, \end{aligned} \quad (4.33)$$

and

$$\begin{aligned} -\infty < \sigma < -a, & \quad \Omega_s^+ - \Omega_s^- = 0, \\ -a < \sigma < 0, & \quad \Omega_s^+ - \Omega_s^- = -2ij\omega h Q(\sigma) e^{j\omega T}, \\ 0 < \sigma < c_0, & \quad \Omega_s^+ + \Omega_s^- = 2g e^{j\omega T}, \\ c_0 < \sigma < \infty, & \quad \Omega_s^+ - \Omega_s^- = 0. \end{aligned} \quad (4.34)$$

According to (3.16), the solution of (4.33) is

$$\Omega_p = \frac{1}{2} \log \frac{[Vt(c+a) - \sqrt{a(t-c)}]}{[Vt(c+a) + \sqrt{a(t-c)}]} + \log \frac{[\sqrt{d(t-c)} - \sqrt{t(d-c)}]}{[\sqrt{d(t-c)} + \sqrt{t(d-c)}]}. \quad (4.35)$$

We now determine the solution of (4.34). By means of the solution of the homogeneous problem  $\Omega_h = \sqrt{t(t-c_0)}$  the Hilbert problem (4.34) can be written as follows:

$$\begin{aligned} -\infty < \sigma < -a, & \quad \left[ \frac{\Omega_s}{\Omega_h} \right]^+ - \left[ \frac{\Omega_s}{\Omega_h} \right]^- = 0, \\ -a < \sigma < 0, & \quad \left[ \frac{\Omega_s}{\Omega_h} \right]^+ - \left[ \frac{\Omega_s}{\Omega_h} \right]^- = -2ij\omega h Q(\sigma) \frac{e^{j\omega T}}{\Omega_h^+}, \\ 0 < \sigma < c_0, & \quad \left[ \frac{\Omega_s}{\Omega_h} \right]^+ - \left[ \frac{\Omega_s}{\Omega_h} \right]^- = \frac{2g e^{j\omega(T-\varphi_0)}}{\Omega_h^+}, \\ c_0 < \sigma < \infty, & \quad \left[ \frac{\Omega_s}{\Omega_h} \right]^+ - \left[ \frac{\Omega_s}{\Omega_h} \right]^- = 0. \end{aligned} \quad (4.36)$$

When  $0 < \sigma < c_0$  we have

$$\varphi_0 = K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]. \quad (4.37)$$

$Q(\sigma)$  is given by

$$Q(\sigma) = \frac{[\sqrt{a(c_0 - \sigma)} + \sqrt{-\sigma(c_0 + a)}]^{\frac{1}{2}}}{[\sqrt{a(c_0 - \sigma)} - \sqrt{-\sigma(c_0 + a)}]^{\frac{1}{2}}} \cdot \frac{[\sqrt{d_0(c_0 - \sigma)} + \sqrt{-\sigma(d_0 - c_0)}]}{[\sqrt{d_0(c_0 - \sigma)} - \sqrt{-\sigma(d_0 - c_0)}]}. \quad (4.38)$$

Using PLEMELJ's formulas (see MUSKHELISHVILI [17]), we get finally

$$\begin{aligned} \Omega_s = & \frac{j\omega h}{\pi} e^{j\omega T} \sqrt{t(t-c_0)} \int_{-a}^0 \frac{Q(\sigma) d\sigma}{\sqrt{-\sigma(c_0 - \sigma)} (\sigma - t)} - \\ & - \frac{g}{\pi} e^{j\omega T} \sqrt{t(t-c_0)} \int_0^{c_0} \frac{e^{-j\omega K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]}}{\sqrt{\sigma(c_0 - \sigma)} (\sigma - t)} d\sigma. \end{aligned} \quad (4.39)$$

Substituting  $c_0 + c_1$  for  $c$  and  $d_0 + d_1$  for  $d$ , expanding and combining the expression obtained with (4.39), we find

$$\begin{aligned}\Omega_1 = & \frac{c_1}{c_0} \sqrt{\frac{t}{t-c_0}} \left[ \frac{1}{2} \sqrt{\frac{a}{c_0+a}} + \sqrt{\frac{d_0}{d_0-c_0}} \right] - \frac{d_1}{d_0} \frac{\sqrt{t(t-c_0)}}{t-d_0} \sqrt{\frac{d_0}{d_0-c_0}} + \\ & + \frac{j\omega h}{\pi} e^{j\omega T} \sqrt{t(t-c_0)} \int_{-a}^0 \frac{Q(\sigma) d\sigma}{\sqrt{-\sigma(c_0-\sigma)} (\sigma-t)} - \\ & - \frac{g}{\pi} e^{j\omega T} \sqrt{t(t-c_0)} \int_0^{c_0} \frac{e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]}}{\sqrt{\sigma(c_0-\sigma)} (\sigma-t)} d\sigma.\end{aligned}\quad (4.40)$$

### 5. Additional conditions

In the expression (4.40) for  $\Omega_1$  three unknown quantities still occur, *viz.*  $g$ ,  $c_1$  and  $d_1$ ;  $g$  is a constant, possibly complex with respect to  $j$ , since we have already anticipated the time behavior of  $G(T-\varphi_0)$ ; the two other unknowns,  $c_1$  and  $d_1$ , are functions of  $T$ , complex with respect to  $j$ , which will appear to be also of exponential type. All three quantities are real with respect to  $i$ . For their determination we have the following three additional conditions.

1. The perturbation velocity must vanish at infinity, *i.e.*,

$$\Omega_1 = 0 \quad \text{as} \quad |t| \rightarrow \infty. \quad (5.1)$$

This condition gives us, as a result of the symmetry of the problem, only one condition real with respect to  $i$ .

2. The perturbation cannot have a source or vortex at infinity. The first part of this condition is equivalent to the requirement that the pressure should remain finite at infinity, which generalizes the closure condition to the unsteady case, for a source at infinity with time-dependent strength gives rise to an infinite pressure there, according to (4.17). The second part is trivially satisfied as a result of the symmetry of the problem. Therefore this second additional condition gives us, in the same way as condition (5.1), only one condition real with respect to  $i$ . As can be seen from (4.24), it is formulated mathematically by

$$\text{res} \Omega_1 = 0 \quad \text{as} \quad |t| \rightarrow \infty. \quad (5.2)$$

3. Finally we have to ensure that the pressure on the cavity is constant not only in space but also in time. This last condition is not expressed by the boundary condition, since the boundary condition (4.28) has been obtained by partial differentiation with respect to  $\varphi_0$ . We shall now formulate Condition 3 mathematically.

The Bernoullian law for unsteady flow has been written in the form (4.17):

$$\frac{\partial \varphi_1}{\partial T} + q_0^2 r_1 + \frac{p_1}{\rho} = D(T). \quad (5.3)$$

We set  $\varphi_1 = 0$  when  $|t| \rightarrow \infty$ . We are allowed to do so, since  $\varphi_1$  is determined only to within a function of time and  $\varphi_1$  is finite at infinity as a result of the absence of a source there. Since at infinity  $r_1 = 0$  and  $p_1 = 0$ , we have  $D(T) = 0$ .

From (4.24) we infer that

$$\varphi_1 = \operatorname{Re} \int_{-\infty}^{\varphi_0} \Omega_1 d\Phi_0 = \operatorname{Re} \int_{-\infty}^t \Omega_1 \frac{d\Phi_0}{dt} dt = K \operatorname{Re} \int_{-\infty}^t \Omega_1 \frac{t-d_0}{t-c_0} dt. \quad (5.4)$$

On the cavity the following identities are valid:

$$q_0 = 1, \quad r_1 = g e^{j\omega(T-\varphi_0)}, \quad p_1 = 0.$$

Thus on the cavity (5.3) can be written as

$$\frac{\partial \varphi_1}{\partial T} = -g e^{j\omega(T-\varphi_0)} \quad (5.5)$$

or, by use of (5.4) and (4.37), as

$$K \operatorname{Re} \int_{-\infty}^{\sigma} \frac{\partial \Omega_1}{\partial T} \frac{t-d_0}{t-c_0} dt = -g e^{j\omega\{T-K[\sigma-(d_0-c_0)\log(c_0-\sigma)]\}}, \quad \text{where } 0 \leq \sigma < c_0. \quad (5.6)$$

This is the mathematical form of the third additional condition. It is easy to prove that this condition is independent of the choice of  $\sigma$  in the above-mentioned interval.

Since the three additional conditions are linear in the unknowns, and since the known members of the equations depend harmonically upon the time, we may put

$$c_1 = \hat{c}_1 e^{j\omega T}, \quad d_1 = \hat{d}_1 e^{j\omega T}.$$

For  $G(T-\varphi_0)$  we have anticipated this time behavior as already remarked.  $\hat{c}_1$  and  $\hat{d}_1$  are real with respect to  $i$  but in general complex with respect to  $j$ .

If we let  $\Omega_1 = \hat{\Omega}_1 e^{j\omega T}$ , (5.6) goes over into

$$\operatorname{Re} K \int_{-\infty}^{\sigma} \hat{\Omega}_1 \frac{t-d_0}{t-c_0} dt = \frac{j}{\omega} g e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]}, \quad 0 \leq \sigma < c_0. \quad (5.7)$$

The path of integration may be deformed according to CAUCHY'S theorem.

We shall now use the three additional conditions in order to determine the three unknown constants,  $\hat{c}_1$ ,  $\hat{d}_1$  and  $g$ . Applying (5.1) to the expression (4.40) for  $\Omega_1$ , we get

$$\begin{aligned} \frac{\hat{c}_1}{c_0} \left[ \frac{1}{2} \sqrt{\frac{a}{c_0+a}} + \sqrt{\frac{d_0}{d_0-c_0}} \right] - \frac{\hat{d}_1}{d_0} \sqrt{\frac{d_0}{d_0-c_0}} - \\ - \frac{j\omega h}{\pi} \int_{-a}^0 Q(\sigma) \frac{d\sigma}{\sqrt{-\sigma(c_0-\sigma)}} + \frac{g}{\pi} \int_0^{c_0} \frac{e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]}}{\sqrt{\sigma(c_0-\sigma)}} d\sigma = 0. \end{aligned} \quad (5.8)$$

Applying (5.2) to (4.40) and using (5.8), we arrive at

$$\begin{aligned} \hat{c}_1 \left[ \frac{1}{2} \sqrt{\frac{a}{c_0+a}} + \sqrt{\frac{d_0}{d_0-c_0}} \right] - \hat{d}_1 \sqrt{\frac{d_0}{d_0-c_0}} - \\ - \frac{j\omega h}{\pi} \int_{-a}^0 Q(\sigma) \frac{\sigma d\sigma}{\sqrt{-\sigma(c_0-\sigma)}} + \frac{g}{\pi} \int_0^{c_0} \frac{e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]}}{\sqrt{\sigma(c_0-\sigma)}} \frac{\sigma d\sigma}{\sqrt{\sigma(c_0-\sigma)}} = 0. \end{aligned} \quad (5.9)$$

From (5.8) and (5.9) we show immediately that

$$\begin{aligned} \hat{c}_1 & \left[ \frac{1}{2} \sqrt{\frac{a}{c_0+a}} + \sqrt{\frac{d_0}{d_0-c_0}} \right] \left( \frac{d_0}{c_0} - 1 \right) \\ &= \frac{j\omega h}{\pi} \int_{-a}^0 Q(\sigma) \frac{(d_0-\sigma)}{\sqrt{-\sigma(c_0-\sigma)}} d\sigma - \frac{g}{\pi} \int_0^{c_0} e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]} \frac{(d_0-\sigma)}{\sqrt{\sigma(c_0-\sigma)}} d\sigma, \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} -\hat{d}_1 & \sqrt{\frac{d_0}{d_0-c_0}} \left( \frac{c_0}{d_0} - 1 \right) \\ &= \frac{j\omega h}{\pi} \int_{-a}^0 Q(\sigma) \frac{(c_0-\sigma)}{\sqrt{-\sigma(c_0-\sigma)}} d\sigma - \frac{g}{\pi} \int_0^{c_0} e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]} \frac{(c_0-\sigma)}{\sqrt{\sigma(c_0-\sigma)}} d\sigma. \end{aligned} \quad (5.11)$$

By means of (5.10) and (5.11) the expression for  $\Omega_1$  can be reduced as follows:

$$\begin{aligned} \hat{\Omega}_1 &= \frac{j\omega h}{\pi} \sqrt{t(t-c_0)} \int_{-a}^0 Q(\sigma) \frac{(d_0-\sigma)(c_0-\sigma)}{\sqrt{-\sigma(c_0-\sigma)}(\sigma-t)(t-c_0)(t-d_0)} d\sigma - \\ & \quad - \frac{g}{\pi} \sqrt{t(t-c_0)} \int_0^{c_0} e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]} \frac{(d_0-\sigma)(c_0-\sigma)}{\sqrt{\sigma(c_0-\sigma)}(\sigma-t)(t-c_0)(t-d_0)} d\sigma. \end{aligned} \quad (5.12)$$

The first integral in (5.12) can be expressed in terms of elementary functions, but since this result is not needed, it is not given here. We are now able to apply Condition 3 as expressed by formula (5.7). We first take  $\sigma=0$  in (5.7) and use formula (5.12) to substitute for  $\hat{\Omega}_1$ . The left-hand member of (5.7) is now replaced by a repeated integral. Since the integrand is a holomorphic function of  $t$  except at the end point  $t=0$ , the integration with respect to  $t$  can be performed along a path which meets the interval  $[-a, c_0]$  of the real axis only at the point  $t=0$ . We now move the end point of the path of integration from  $t=0$  to a neighboring point  $t=\varepsilon$ , with  $\text{Im } \varepsilon > 0$ . Since both integrals in the repeated integral are now absolutely convergent, we can change the order of integration. Evaluating the inner integral and afterwards letting  $\varepsilon \rightarrow 0$ , which is legitimate on account of the uniform convergence of the outer integral, we get

$$\begin{aligned} K \frac{j\omega h}{\pi} & \left[ -2 \int_{-a}^0 Q(\sigma) \frac{d_0-\sigma}{\sqrt{-\sigma(c_0-\sigma)}} d\sigma - \int_{-a}^0 Q(\sigma) \frac{d_0-\sigma}{c_0-\sigma} \log \frac{[\sqrt{c_0-\sigma}-\sqrt{-\sigma}]}{[\sqrt{c_0-\sigma}+\sqrt{-\sigma}]} d\sigma \right] - \\ & - \frac{Kg}{\pi} \left[ -2 \int_0^{c_0} e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]} \frac{(d_0-\sigma)}{\sqrt{\sigma(c_0-\sigma)}} d\sigma + \right. \\ & \quad \left. + \int_0^{c_0} e^{-j\omega K[\sigma-(d_0-c_0)\log(c_0-\sigma)]} \left[ \arccos \left( \frac{2\sigma}{c_0} - 1 \right) \right] \frac{d_0-\sigma}{c_0-\sigma} d\sigma \right] \\ &= \frac{jg}{\omega} e^{j\omega K(d_0-c_0)\log c_0}. \end{aligned} \quad (5.13)$$

Use has been made of the following identities:

$$\text{Re} \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\varepsilon} \frac{\sqrt{t(t-c_0)} dt}{(t-c_0)^2 (\sigma-t)} = \frac{2}{\sigma-c_0} - \sqrt{\frac{-\sigma}{c_0-\sigma}} \frac{1}{c_0-\sigma} \log \frac{[\sqrt{c_0-\sigma}-\sqrt{-\sigma}]}{[\sqrt{c_0-\sigma}+\sqrt{-\sigma}]} \quad \text{if } \sigma \leq 0,$$

and

$$= \frac{2}{c_0 - \sigma} - \left[ \arccos \left( \frac{2\sigma}{c_0} - 1 \right) \right] \sqrt{\frac{\sigma}{c_0 - \sigma}} \frac{1}{c_0 - \sigma} \quad \text{if } 0 \leq \sigma \leq c_0. \quad (5.13a)$$

*Remark.* This expression is finite for  $\sigma = c_0$ . The last integral in (5.13) can be reduced as follows:

$$\begin{aligned} & \int_0^{c_0} e^{-j\omega K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]} \left[ \arccos \left( \frac{2\sigma}{c_0} - 1 \right) \right] \frac{d_0 - \sigma}{c_0 - \sigma} d\sigma \\ &= \frac{j}{\omega K} \int_0^{c_0} \left[ \arccos \left( \frac{2\sigma}{c_0} - 1 \right) \right] d e^{-j\omega K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]} \\ &= -\frac{\pi j}{\omega K} e^{j\omega K(d_0 - c_0) \log c_0} + \frac{j}{\omega K} \int_0^{c_0} \frac{e^{-j\omega K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]}}{\sqrt{\sigma(c_0 - \sigma)}} d\sigma. \end{aligned} \quad (5.14)$$

The integral occurring in the last member of (5.14) and the integral of the same type in (5.13) can be expressed in confluent hypergeometric functions.

Using (5.14), we can write (5.13) finally as

$$\begin{aligned} & -j\omega h \left[ 2 \int_{-a}^0 Q(\sigma) \frac{d_0 - \sigma}{\sqrt{-\sigma(c_0 - \sigma)}} d\sigma + \int_{-a}^0 Q(\sigma) \frac{d_0 - \sigma}{c_0 - \sigma} \log \frac{[c_0 - \sigma - 1 - \sigma]}{[c_0 - \sigma + 1 - \sigma]} d\sigma \right] + \\ & + g \int_0^{c_0} e^{-j\omega K[\sigma - (d_0 - c_0) \log(c_0 - \sigma)]} \frac{1}{\sqrt{\sigma(c_0 - \sigma)}} \cdot \left[ 2(d_0 - \sigma) - \frac{j}{\omega K} \right] d\sigma = 0. \end{aligned} \quad (5.15)$$

Formula (5.15) gives us  $g$ ;  $\hat{c}_1$  and  $\hat{d}_1$  are then found from (5.10) and (5.14). The solution is now completely determined.

### 6. First order approximation for small breadth

$\alpha$ ) In this section "small" is understood to mean small with respect to the length of the cavity. We start by determining the first-order approximation for the steady flow. For that purpose we fix the value of  $K$  at 1 and consider the quantity  $a$ , which has so far been an arbitrary constant, as an unknown to be determined from the condition (3.24) of Section 3. Combination of this formula (3.24) with (3.25) and (3.26) or (3.27) gives us the relation between  $a$  on the one hand and  $b$  and  $\sigma$  on the other hand. For the computation it is convenient to consider  $a$  and  $c$  as given and to determine  $b$  and  $\sigma$  as functions of  $a$  and  $c$ . We suppose  $a$  to be small. It will appear then that for fixed  $c \neq 0$  also  $b$  is small; i.e., as  $a$  tends to zero,  $b$  also tends to zero. The cavitation number  $\sigma$  as a function of  $a$  and  $c$  is given by (3.25). We expand this expression in powers of  $\sqrt{\frac{a}{c}}$  so that

$$1 + \sigma = 1 + 4 \sqrt{\frac{a}{c}} + \dots \quad (6.1)$$

Hence we have to a first order of approximation

$$\sigma = 4 \sqrt{\frac{a}{c}}. \quad (6.2)$$

We now expand the integral in (3.24) in powers of  $\sqrt{\frac{a}{c}}$ , using (3.26) and (3.27). For that purpose we introduce first a new variable of integration and take the real part of the integral.

$$\begin{aligned} & \int_{-a}^0 \frac{[\sqrt{a(c-\sigma)} + \sqrt{-\sigma(c+a)}]^{\frac{1}{2}} [\sqrt{d(c-\sigma)} + \sqrt{-\sigma(d-c)}]}{[\sqrt{a(c-\sigma)} - \sqrt{-\sigma(c+a)}]^{\frac{1}{2}} [\sqrt{d(c-\sigma)} - \sqrt{-\sigma(d-c)}]} \frac{\sigma-d}{\sigma-c} d\sigma \\ &= \int_0^1 y^{-\frac{1}{2}} \frac{d}{c} \left[ 1 + \sqrt{\frac{a(d-c)}{d(c+a)}} \frac{1-y}{1+y} \right]^2 \frac{4ac(c+a)(1-y^2)}{\{c(1+y)^2 + 4ay\}^2} dy, \end{aligned} \quad (6.3)$$

where the new variable of integration  $y$  is given by

$$y = \frac{\sqrt{a(c-\sigma)} - \sqrt{-\sigma(c+a)}}{\sqrt{a(c-\sigma)} + \sqrt{-\sigma(c+a)}}. \quad (6.3a)$$

According to (3.26) and (3.27) we have to a first order of approximation

$$\sqrt{\frac{d-c}{d}} = \frac{1}{2} \sqrt{\frac{a}{c}}$$

and

$$d - c = 0. \quad (6.4)$$

Hence to a second order of approximation the expression in (6.3) becomes equal to

$$4a \int_0^1 y^{-\frac{1}{2}} \frac{1-y}{(1+y)^3} dy = \frac{a}{2} (\pi + 4). \quad (6.5)$$

From this formula it is seen immediately that to the same order of approximation

$$b = \frac{1}{2} a (\pi + 4). \quad (6.6)$$

Expanding the expression (3.17) for  $w$  into ascending powers of  $\sqrt{\frac{a}{c}}$ , we get

$$w = 1 - \sqrt{\frac{a}{c}} \left[ \sqrt{\frac{t-c}{t}} + \sqrt{\frac{t}{t-c}} \right] + \dots \quad (6.7)$$

Substitution of (6.2) gives us

$$w = 1 - \frac{\sigma}{4} \left[ \sqrt{\frac{t-c}{t}} + \sqrt{\frac{t}{t-c}} \right] + \dots \quad (6.8)$$

These expansions are uniformly valid in the interior of the cut  $t$ -plane. It may easily be seen that to a first order of approximation

$$dz = dt. \quad (6.9)$$

Hence to that approximation  $w$  is given by

$$w = 1 - \frac{\sigma}{4} \left[ \sqrt{\frac{z-c}{z}} + \sqrt{\frac{z}{z-c}} \right]. \quad (6.10)$$

The relation between  $b$ ,  $c$  and  $\sigma$  is

$$\sigma = 4 \sqrt{\frac{2b}{(\pi+4)c}}. \quad (6.11)$$

In the neighborhood of  $z=0$  the behavior of  $w$  is given by

$$w = - \sqrt{\frac{2b}{\pi+4}} (-z)^{-\frac{1}{2}} + \dots. \quad (6.12)$$

The quantity  $c$  may be considered as a first-order approximation to the cavity length.

$\beta$ ) First-order approximation for the unsteady motion. We determine the first-order approximation to the unsteady perturbation field for small breadth of the plate by expanding  $\hat{\Omega}_1$  in a series of ascending powers of  $\sqrt{\frac{a}{c_0}}$ . From (5.12), by again introducing  $y$  as a new variable of integration, we find that to a first order of approximation

$$\begin{aligned} \hat{\Omega}_1 &= - \frac{j\omega h}{\pi} 4 \sqrt{\frac{a}{c_0}} c_0^2 \frac{1}{\sqrt{t(t-c_0)}(t-c_0)} \int_0^1 \frac{y^{-\frac{1}{2}} dy}{(1+y)^2} - \\ &\quad - \frac{g}{\pi} \sqrt{\frac{t}{t-c_0}} \frac{1}{t-c_0} \int_0^{c_0} e^{-j\omega\sigma} \sqrt{\frac{c_0-\sigma}{\sigma}} \frac{c_0-\sigma}{\sigma-t} d\sigma \\ &= -j\omega h \frac{\pi+2}{\pi} c_0^2 \sqrt{\frac{a}{c_0}} \frac{1}{\sqrt{t(t-c_0)}(t-c_0)} - \\ &\quad - \frac{g}{\pi} \sqrt{\frac{t}{t-c_0}} \frac{1}{t-c_0} \int_0^{c_0} e^{-j\omega\sigma} \sqrt{\frac{c_0-\sigma}{\sigma}} \frac{c_0-\sigma}{\sigma-t} d\sigma. \end{aligned} \quad (6.13)$$

Relation (5.15) connecting  $\omega h$  and  $g$  is equivalent, to the same order of approximation, to

$$-2j\omega h(\pi+2) \sqrt{\frac{a}{c_0}} c_0 + g \int_0^{c_0} e^{-j\omega\sigma} \frac{2(c_0-\sigma) - \frac{j}{\omega}}{\sqrt{\sigma(c_0-\sigma)}} d\sigma = 0. \quad (6.14)$$

Formula (5.10) becomes

$$\hat{c}_1 \frac{1}{2} \sqrt{\frac{a}{c_0}} = \frac{j\omega h}{\pi} (\pi+2) \sqrt{\frac{a}{c_0}} c_0 - \frac{g}{\pi} \int_0^{c_0} e^{-j\omega\sigma} \sqrt{\frac{c_0-\sigma}{\sigma}} d\sigma. \quad (6.15)$$

The integrals in (6.14) and (6.15) can be expressed in terms of cylindrical functions by means of the following identities:

$$\int_0^{c_0} \frac{e^{-j\omega\sigma} d\sigma}{\sqrt{\sigma(c_0-\sigma)}} = \pi e^{-j\omega \frac{c_0}{2}} J_0\left(\omega \frac{c_0}{2}\right), \quad (6.16)$$

and

$$\int_0^{c_0} e^{-j\omega\sigma} \sqrt{\frac{c_0-\sigma}{\sigma}} d\sigma = \frac{\pi c_0}{2} e^{-j\omega \frac{c_0}{2}} \left[ J_0\left(\omega \frac{c_0}{2}\right) + j J_1\left(\omega \frac{c_0}{2}\right) \right]. \quad (6.17)$$

Hence (6.14) may be written as

$$-2j\omega h \frac{\pi+2}{\pi} \sqrt{\frac{a}{c_0}} + g e^{-j\omega \frac{c_0}{2}} \left[ J_0\left(\omega \frac{c_0}{2}\right) \left(1 - \frac{j}{\omega c_0}\right) + j J_1\left(\omega \frac{c_0}{2}\right) \right] = 0 \quad (6.18)$$

and (6.15) as

$$\frac{\hat{c}_1}{c_0} \sqrt{\frac{a}{c_0}} = 2j\omega h \frac{\pi+2}{\pi} \sqrt{\frac{a}{c_0}} - g e^{-j\omega \frac{c_0}{2}} \left[ J_0\left(\omega \frac{c_0}{2}\right) + j J_1\left(\omega \frac{c_0}{2}\right) \right]. \quad (6.19)$$

From (6.18) and (6.19) we show immediately that

$$g = 2j\omega h \frac{\pi+2}{\pi} \sqrt{\frac{2b}{(\pi+4)c_0}} e^{j\omega \frac{c_0}{2}} J_0\left(\omega \frac{c_0}{2}\right) \left(1 - \frac{j}{\omega c_0}\right) + j J_1\left(\omega \frac{c_0}{2}\right) \Big]^{-1} \quad (6.20)$$

and

$$\omega \hat{c}_1 = 2\omega h \frac{\pi+2}{\pi} J_0\left(\omega \frac{c_0}{2}\right) \left[ J_0\left(\omega \frac{c_0}{2}\right) \left(1 - \frac{j}{\omega c_0}\right) + j J_1\left(\omega \frac{c_0}{2}\right) \right]^{-1}. \quad (6.21)$$

In the derivation use has been made of (6.6) in order to eliminate  $a$ . For  $\omega \hat{d}_1$  we find the same expression as for  $\omega \hat{c}_1$ . The relation between  $\omega \hat{c}_1$  and  $g$  reads

$$j\omega \hat{c}_1 \sqrt{\frac{2b}{(\pi+4)c_0}} = g e^{-j\omega \frac{c_0}{2}} J_0\left(\omega \frac{c_0}{2}\right) \quad (6.22)$$

or

$$j\omega \hat{c}_1 \frac{\sigma}{4} = g e^{-j\omega \frac{c_0}{2}} J_0\left(\omega \frac{c_0}{2}\right).$$

We shall now discuss the conditions to be imposed on the parameters  $\omega$ ,  $h$  and  $b$ , in order that the approximations introduced so far be valid.

Remembering the discussion at the beginning of Section 4 and maintaining the notation introduced there, we must first require that

$$\varepsilon_1 = \frac{\omega h}{q_c} \ll 1, \quad (6.23)$$

$$\varepsilon_2 = \frac{h}{b} \ll 1, \quad (6.24)$$

$$\varepsilon_3 = \frac{\varepsilon_1}{\varepsilon_2} \gg 1. \quad (6.25)$$

As a result of the approximation for small breadth  $2b$  of the flat plate, we must further require that

$$\varepsilon_4 = \sqrt{\frac{a}{c_0}} \ll 1. \quad (6.26)$$

However, to be sure that the unsteady effects are not of the same order of magnitude as the second-order steady effects, we must impose the additional condition that

$$\sqrt{\frac{a}{c_0}} \ll \frac{\omega h}{q_c}, \quad (6.27)$$

or, in terms of the smallness parameters  $\varepsilon_1$  and  $\varepsilon_2$ , that

$$\varepsilon_4 \ll \varepsilon_1. \quad (6.28)$$

The five conditions (6.23) to (6.27) are not independent. An independent set of conditions is given by

$$\varepsilon_1 = \frac{\omega h}{q_c} \ll 1, \quad (6.23)$$

$$\varepsilon_3 = \frac{\omega b}{q_c} \gg 1, \quad (6.25)$$

$$\varepsilon_4 = \sqrt{\frac{a}{c_0}} \ll \varepsilon_1. \quad (6.28)$$

Using (6.6), we may replace (6.28) by

$$\varepsilon_4 = \sqrt{\frac{2b}{(\pi+4)c_0}} \ll \varepsilon_1. \quad (6.29)$$

From these conditions it can be shown easily that

$$\max \left\{ \frac{1}{h} \sqrt{\frac{b}{c_0}}, \frac{1}{b} \right\} \ll \frac{\omega}{q_c} \ll \frac{1}{h}, \quad (6.30)$$

and

$$h \ll b \ll c_0. \quad (6.31)$$

In the case that the approximation for small breadth is not carried through, the conditions (6.30) and (6.31) have to be replaced by

$$\frac{1}{b} \ll \frac{\omega}{q_c} \ll \frac{1}{h}, \quad (6.32)$$

and

$$h \ll b. \quad (6.33)$$

Returning to formula (6.21) and taking into account that according to (6.30)  $\omega \gg 1$ , we conclude that the velocity of the moving rear end of the cavity is nearly in phase with the velocity of the flat plate. The deviation of the rear end of the cavity from its steady position is of the same order of magnitude as the deviation of the flat plate.

## Part II. Linearized theory of unsteady cavity flow

### 7. Summary

In the linearized theory to be discussed in the following sections, the vertical flat plate is replaced by a singularity. It will appear that the velocity field of this singularity in the steady case is identical with the field given by (6.10), *i.e.*, by the first-order term of (3.17). The strength of the singularity can then be expressed in terms of half the breadth  $b$  of the flat plate.

In the unsteady case the strength of the singularity is assumed to vary with time. Considering the variations in this strength to be small compared to the steady part of it (second linearization), we may proceed in a manner analogous to that for the steady case. It will appear that in the case of harmonic time-dependence with circular frequency  $\omega$  the unsteady part of the flow field can be identified with the first order term of  $\hat{\Omega}_1$ , as derived in Section 6 $\beta$ , if  $\omega$  is assumed to be large and the amplitude of the unsteady part of the strength of the singularity is put equal to

$$-j\omega h \frac{\pi+2}{\pi} \sqrt{\frac{2b}{\pi+4}}.$$

A treatment of the same problem by means of the acceleration-potential requires the application of additional conditions, different from the ones used for the method of the complex velocity. It is shown in Section 11 that both methods are equivalent, provided the right conditions are applied.

Finally we give an expression for the drag in the case when our linearizing assumptions are valid.

### 8. Linearized theory of steady flow

We suppose that the flow pattern as a result of the presence of the flat plate can be considered as a small perturbation of the original uniform flow field. This will generally be the case if the breadth  $2b$  of the plate is small with respect to the cavity length. The plate can then be replaced by a singularity. The rear end of the cavity will show a similar singularity. Within the frame work of linearized theory the occurrence of these singularities can be inferred from the presence of a stagnation point in the neighborhood. We shall not go into the details but refer to [18].

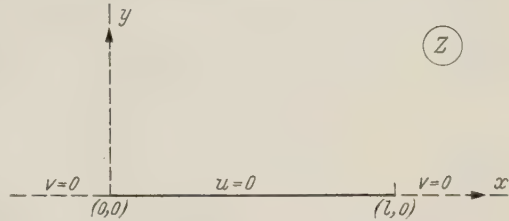


Fig. 4. Linearized boundary-value problem

We choose the Cartesian coordinate system in such a way that the cavity extends between the points  $(0, 0)$  and  $(l, 0)$ . The velocity at infinity has the direction of the positive  $x$ -axis. The units of length and time are so chosen that the constant velocity  $q_c$  along the cavity is equal to 1. We put  $q = (1 + u, v)$ . Since  $1 - U \ll 1$ , we have  $u \ll 1$  and  $v \ll 1$ . We shall neglect second and higher powers of  $u$  and  $v$  and their products (linearization).  $w = u - iv$  is a holomorphic function of  $z = x + iy$ .

The linearized Bernoullian law reads

$$p - p_c = -\rho u. \quad (8.1)$$

Hence we show that

$$u_\infty = -\frac{1}{2}\sigma. \quad (8.2)$$

On the cavity,

$$u = 0. \quad (8.3)$$

Since the maximum width of the cavity is of the same order of magnitude as the perturbation velocities, we may apply condition (8.3) on the  $x$ -axis between  $(0, 0)$  and  $(l, 0)$  instead of on the cavity. The holomorphic function  $w$  now must satisfy the following mixed boundary-value problem (see Figure 4): Determine  $w(z)$ , holomorphic in the upper half of the  $z$ -plane, such that it assumes the following boundary values on the real axis:

$$\begin{aligned} -\infty < x < 0, \quad v = 0, \quad \text{or} \quad \text{Im } w = 0, \\ 0 < x < l, \quad u = 0, \quad \text{or} \quad \text{Re } w = 0, \\ l < x < \infty, \quad v = 0, \quad \text{or} \quad \text{Im } w = 0. \end{aligned} \quad (8.4)$$

By means of the Schwarzian reflection principle this boundary-value problem for  $w(z)$  can be transformed into a Hilbert problem with the following linear relations along the real axis:

$$\begin{aligned} -\infty < x < 0, \quad w^+ - w^- &= 0, \\ 0 < x < l, \quad w^+ + w^- &= 0, \\ l < x < \infty, \quad w^+ - w^- &= 0. \end{aligned} \quad (8.5)$$

The general solution with integrable singularities at  $z=0$  and  $z=l$  is

$$w = A \sqrt{\frac{z-l}{z}} + B \sqrt{\frac{z}{z-l}}. \quad (8.6)$$

The cut for both square roots is made from 0 to  $l$ ; the branches are so chosen that

$$\sqrt{\frac{z-l}{z}} \sim 1$$

and

$$\sqrt{\frac{z}{z-l}} \sim 1 \quad \text{as } |z| \rightarrow \infty. \quad (8.7)$$

We assume that the strength of the singularity at  $z=0$ , determined by the flat plate, is given. There we have

$$w \sim A \sqrt{l} \frac{1}{\sqrt{-z}}. \quad (8.8)$$

Hence the strength of the singularity  $= A \sqrt{l}$ . This gives us one relation between  $A$  and  $l$ . For the determination of the unknowns  $A$ ,  $B$  and  $l$  we still need two relations. These are furnished by the following additional conditions:

$$1. \quad w = -\frac{1}{2}\sigma \quad \text{at } z = \infty, \quad \text{according to (8.2),} \quad (8.9)$$

$$2. \quad \text{res. } w = 0 \quad \text{at } z = \infty. \quad (8.10)$$

This condition is equivalent to the requirement that the cavity be closed (see [18]). From (8.9) we deduce

$$A + B = -\frac{1}{2}\sigma, \quad (8.11)$$

and from (8.10)

$$A - B = 0. \quad (8.12)$$

Hence

$$A = B = -\frac{1}{4}\sigma. \quad (8.13)$$

Thus for  $w$  we have the formula

$$w = -\frac{\sigma}{4} \left[ \sqrt{\frac{z-l}{z}} + \sqrt{\frac{z}{z-l}} \right]. \quad (8.14)$$

This expression agrees with (6.10) if we identify  $c$  and  $l$ . From (6.9) it follows that

$$A \sqrt{l} = -\sqrt{\frac{2b}{\pi+4}}. \quad (8.15)$$

This is the connection between the strength of the singularity at  $z=0$  and the breadth  $2b$  of the plate. We have shown now that linearized theory can be understood as a first order approximation of reentrant-jet theory. The same statement holds for Riabouchinsky flow (see [13]).

### 9. Unsteady flow; method of the complex velocity

The coordinate system and the units of length and time are chosen as in Section 8. It is assumed that the strength of the singularity at  $z=0$  will vary harmonically with circular frequency  $\omega$ , and that the amplitude of the time-dependent part is small with respect to the time-independent part, but large with respect to second and higher powers of this latter. This last assumption enables us to treat the unsteady effects as linear perturbations of the linearized steady flow (second linearization). Hence all quantities can be split up additively into steady and unsteady parts. Accordingly we put

$$\begin{aligned} u &= u_0 + u_1 & \text{with } u_1 &= \hat{u}_1 e^{j\omega t}, \\ v &= v_0 + v_1, & v_1 &= \hat{v}_1 e^{j\omega t}, \\ A &= A_0 + A_1, & A_1 &= \hat{A}_1 e^{j\omega t}, \\ B &= B_0 + B_1, & B_1 &= \hat{B}_1 e^{j\omega t}, \\ l &= l_0 + l_1, & l_1 &= \hat{l}_1 e^{j\omega t}, \\ p &= p_0 + p_1, & p_1 &= \hat{p}_1 e^{j\omega t}. \end{aligned} \tag{9.4}$$

*Remark.* The time variable is denoted as usual by  $t$  instead of by  $T$ , differently from the notation used in Part I. As a result of the unsteady motion the boundary condition on the cavity will change. For its derivation we start from the linearized Euler equations of motion for an ideal fluid:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= -\frac{1}{\varrho} \frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} &= -\frac{1}{\varrho} \frac{\partial p}{\partial y}. \end{aligned} \tag{9.2}$$

On the cavity we have  $p=p_c$ ; thus  $\frac{\partial p}{\partial x}=0$ , and by (9.2) we get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{9.3}$$

By use of (9.1) we obtain

$$j\omega \hat{u}_1 + \frac{\partial \hat{u}_1}{\partial x} = 0. \tag{9.4}$$

The general solution of this differential equation is

$$\hat{u}_1 = \hat{g} e^{-j\omega x}, \tag{9.5}$$

where  $\hat{g}$  is an arbitrary constant, real with respect to  $i$ , but possibly complex with respect to  $j$ .

For  $w = w_0 + w_1$  we now have the following mixed boundary-value problem: Determine  $w(z)$ , holomorphic in the upper half of the  $z$ -plane, so as to satisfy the following boundary values on the real axis:

$$\begin{aligned} -\infty < x < 0, & \quad v = 0, & \quad \text{or} \quad \text{Im } w = 0, \\ 0 < x < l, & \quad u = \hat{g} e^{j\omega(t-x)}, & \quad \text{or} \quad \text{Re } w = \hat{g} e^{j\omega(t-x)}, \\ l < x < \infty, & \quad v = 0, & \quad \text{or} \quad \text{Im } w = 0. \end{aligned} \quad (9.6)$$

By means of the Schwarzian reflection principle this boundary-value problem can be transformed into a Hilbert problem with the following linear relations along the real axis:

$$\begin{aligned} -\infty < x < 0, & \quad w^+ - w^- = 0, \\ 0 < x < l, & \quad w^+ + w^- = 2\hat{g} e^{j\omega(t-x)}, \\ l < x < \infty, & \quad w^+ - w^- = 0. \end{aligned} \quad (9.7)$$

The solution is

$$w = -\frac{\hat{g}}{\pi} e^{j\omega t} \sqrt{z(z-l)} \int_0^l \frac{e^{-j\omega x} dx}{\sqrt{x(l-x)}(x-z)} + A \sqrt{\frac{z-l}{z}} + B \sqrt{\frac{z}{z-l}}. \quad (9.8)$$

We now divide this expression for  $w$  into its steady part  $w_0$  and its unsteady part  $w_1$ . We find that

$$w_0 = A_0 \sqrt{\frac{z-l_0}{z}} + B_0 \sqrt{\frac{z}{z-l_0}}, \quad (9.9)$$

$$\begin{aligned} w_1 = & -\frac{\hat{g}}{\pi} e^{j\omega t} \sqrt{z(z-l_0)} \int_0^{l_0} \frac{e^{-j\omega x} dx}{\sqrt{x(l_0-x)}(x-z)} - \\ & - A_0 \frac{l_1}{2} \frac{1}{\sqrt{z(z-l_0)}} + B_0 \frac{l_1}{2} \sqrt{\frac{z}{z-l_0}} \frac{1}{z-l_0} + A_1 \sqrt{\frac{z-l_0}{z}} + B_1 \sqrt{\frac{z}{z-l_0}}. \end{aligned} \quad (9.10)$$

For the relation between  $A_0$  and the breadth  $2b$  of the flat plate, the determination of  $B_0$ , and the connection between  $l_0$  and the cavitation number  $\sigma$ , we refer to Section 8.

Assuming that in (9.10) the strength of the singularity at  $z=0$  is given, we still need three other conditions for the determination of the four unknowns,  $\hat{g}$ ,  $\hat{l}_1$ ,  $\hat{A}_1$  and  $\hat{B}_1$ . These conditions read as follows:

1. As in Section 8 we must have

$$\begin{aligned} w = -\frac{1}{2}\sigma & \quad \text{at } z = \infty, \\ \text{or} & \\ w_1 = 0 & \quad \text{at } z = \infty. \end{aligned} \quad (9.11)$$

This condition means that the perturbation velocities vanish at infinity.

2. res.  $w = 0$  at  $z = \infty$ ,  
or, according to Section 8,

$$\text{res. } w_1 = 0 \quad \text{at } z = \infty. \quad (9.12)$$

This condition means that there is no source or sink at infinity. We already know that there is no vortex at infinity as a result of the symmetry of the problem.

As a result of this condition the pressure remains finite at infinity in the unsteady case too.

3. Boundary condition (9.5) on the cavity has been obtained by considering the derivative of the pressure  $p$  along the boundary of the cavity. Hence it is possible that a time-dependent additive function, a constant being included as a special case, is not taken into account for the pressure in the cavity. In any case it is necessary to connect the pressure on the cavity with the pressure at infinity. This is the third condition. We now formulate it mathematically. According to (9.2) we have

$$\frac{\partial p_0}{\partial x} = -\varrho \frac{\partial u_0}{\partial x},$$

and

$$\frac{\partial p_1}{\partial x} = -\varrho \left( j\omega u_1 + \frac{\partial u_1}{\partial x} \right). \quad (9.13)$$

The unsteady motion is not allowed to disturb the connection between the pressure at infinity and the pressure on the cavity, *i.e.*, we require that

$$p_{1,c} - p_{1,\infty} = 0, \quad (9.14)$$

or, by virtue of (9.13), that

$$\int_{-\infty}^{+0} \left( j\omega \hat{u}_1 + \frac{\partial \hat{u}_1}{\partial x} \right) dx = 0. \quad (9.15)$$

Integrating, we obtain

$$j\omega \int_{-\infty}^{+0} \hat{u}_1 dx = -\hat{u}_1(+0, 0) = -\hat{g}, \quad (9.16)$$

or

$$\int_{-\infty}^{+0} \hat{u}_1 dx = \frac{j\hat{g}}{\omega}. \quad (9.17)$$

Application of (9.11) and (9.12) gives

$$\frac{\hat{g}}{\pi} \int_0^{l_0} \frac{e^{-j\omega x} dx}{\sqrt{x(l_0-x)}} + \hat{A}_1 + \hat{B}_1 = 0, \quad (9.18)$$

$$\begin{aligned} \frac{\hat{g}}{\pi} \left[ -\frac{l_0}{2} \int_0^{l_0} \frac{e^{-j\omega x} dx}{\sqrt{x(l_0-x)}} + \int_0^{l_0} \frac{e^{-j\omega x} x dx}{\sqrt{x(l_0-x)}} \right] - \\ - A_0 \frac{\hat{l}_1}{2} - \hat{A}_1 \frac{l_0}{2} + B_0 \frac{\hat{l}_1}{2} + \hat{B}_1 \frac{l_0}{2} = 0. \end{aligned} \quad (9.19)$$

Since  $A_0 = B_0 = -\frac{1}{4}\sigma$ , it follows that

$$\begin{aligned} \frac{\hat{g}}{\pi} \int_0^{l_0} e^{-j\omega x} \sqrt{\frac{x}{l_0-x}} dx + \hat{B}_1 l_0 = 0, \\ \frac{\hat{g}}{\pi} \int_0^{l_0} e^{-j\omega x} \sqrt{\frac{l_0-x}{x}} dx + \hat{A}_1 l_0 = 0. \end{aligned} \quad (9.20)$$

By means of (9.10) and (9.20) we find that

$$\hat{u}_1 = -\frac{\hat{g}}{\pi} \frac{1}{V-x(l_0-x)} \int_0^{l_0} e^{-j\omega\xi} \frac{\sqrt{\xi(l_0-\xi)}}{\xi-x} d\xi - \frac{\sigma \hat{l}_1}{8} \frac{l_0}{(-x)^{\frac{1}{2}}(l_0-x)^{\frac{3}{2}}}. \quad (9.21)$$

We can now apply (9.17)

$$\begin{aligned} & -\frac{\hat{g}}{\pi} \int_0^{l_0} e^{-j\omega\xi} \sqrt{\xi(l_0-\xi)} d\xi \int_{-\infty}^0 \frac{dx}{V-x(l_0-x)(\xi-x)} - \\ & -\frac{\sigma \hat{l}_1 l_0}{8} \int_{-\infty}^0 \frac{dx}{(-x)^{\frac{1}{2}}(l_0-x)^{\frac{3}{2}}} = \frac{jg}{\omega}. \end{aligned} \quad (9.22)$$

The inversion of the order of integration can be justified in the same way as in Section 5. We now use the following identities:

$$\begin{aligned} & \int_{-\infty}^0 \frac{dx}{V-x(l_0-x)(\xi-x)} = \frac{1}{V\xi(l_0-\xi)} \left[ \arccos\left(\frac{2\xi}{l_0} - 1\right) \right], \\ & \int_{-\infty}^0 \frac{dx}{(-x)^{\frac{1}{2}}(l_0-x)^{\frac{3}{2}}} = \frac{2}{l_0}. \end{aligned} \quad (9.23)$$

Integrating partially, from (9.22) we get

$$-\frac{\hat{g}}{\pi} \int_0^{l_0} \frac{e^{-j\omega x} dx}{Vx(l_0-x)} + \frac{\sigma \hat{l}_1 j \omega}{4} = 0. \quad (9.24)$$

By means of (6.16) the integral in this formula can be expressed in terms of a Bessel function:

$$\frac{1}{4} \sigma \hat{l}_1 j \omega = \hat{g} e^{-\frac{1}{2}j\omega l_0} J_0\left(\omega \frac{l_0}{2}\right). \quad (9.25)$$

*Remark.* It is at once possible to write the solution (9.8) of the boundary-value problem (9.6) in the form

$$w = \frac{\hat{g}}{\pi} e^{j\omega t} \frac{1}{Vz(z-l)} \int_0^l \frac{e^{-j\omega x} Vx(l-x) dx}{x-z} + A' \sqrt{\frac{z-l}{z}} + B' \sqrt{\frac{z}{z-l}}. \quad (9.26)$$

By putting  $A' = A_0$  and  $B' = B_0$ , the conditions (9.11) and (9.12) are automatically satisfied. Finally it is possible to get from (9.22) to (9.24) without evaluating the first integral in (9.23). In this case one uses the fact that the integral can be considered as the solution of a certain boundary-value problem.

### 10. Linearized unsteady cavity flow as a first order approximation

We shall derive expressions for  $\hat{\Omega}_1$  and  $\hat{w}_1$  which can be compared in order to show that, for large  $\omega$ ,  $\hat{w}_1$  can be identified with the first-order approximation to  $\hat{\Omega}_1$ , derived in Section 6 $\beta$ . For that purpose we express  $\hat{\Omega}_1$  in terms of  $g$  and  $\hat{w}_1$  in terms of  $\hat{g}$ . From (6.13) and (6.14) we find that

$$\hat{\Omega}_1 = \frac{g}{\pi} \frac{1}{Vt(t-c_0)} - \int_0^{c_0} e^{-j\omega\sigma} \frac{V\sigma(c_0-\sigma)}{\sigma-t} d\sigma + c_0 \frac{j}{2\omega} \frac{g}{\pi} \frac{1}{t^{\frac{1}{2}}(t-c_0)^{\frac{3}{2}}} \int_0^{c_0} \frac{e^{-j\omega\sigma} d\sigma}{V\sigma(c_0-\sigma)}. \quad (10.1)$$

It has been shown that this expression is valid only for large  $\omega$ . Combination of (9.10), (9.20) and (9.24) gives us

$$\hat{w}_1 = \frac{\hat{g}}{\pi} \frac{1}{V_z(z-l_0)} \int_0^{l_0} e^{-j\omega\xi} \frac{V_\xi(l_0-\xi)}{\xi-z} d\xi + l_0 \frac{j}{2\omega} \frac{\hat{g}}{\pi} \frac{1}{z^{\frac{1}{2}}(z-l_0)^{\frac{3}{2}}} \int_0^{l_0} \frac{e^{-j\omega\xi} d\xi}{V_\xi(l_0-\xi)}. \quad (10.2)$$

In Section 8 we have identified the following quantities, viz,  $t$  and  $z$ ,  $c_0$  and  $l_0$ . If we may now identify  $g$  and  $\hat{g}$  also, the two expressions (10.1) and (10.2) become the same. If we put the strength of the singularity for  $\hat{w}_1$  at  $z=0$  equal to the strength of the singularity for  $\hat{\Omega}_1$  at  $t=0$ , we infer from both formulas that indeed  $g$  and  $\hat{g}$  must be identified. This gives the proof of the statement that for the unsteady case, too, linearized cavity theory is a first-order approximation of reentrant-jet theory for large  $\omega$ . From (6.22) and (9.25) we see also that  $\hat{c}_1$  and  $\hat{l}_1$  must be identified.

Finally we derive an expression for the strength of the singularity at  $z=0$  in terms of  $h$  and  $\omega$ . This expression can most easily be derived from (6.13). According to that formula  $\hat{\Omega}_1$  has the following behavior near  $t=0$ :

$$\hat{\Omega}_1 \sim -j\omega h \frac{\pi+2}{\pi} \sqrt{a} (-t)^{-\frac{1}{2}}, \quad (10.3)$$

where  $(-t)^{-\frac{1}{2}}$  is positive for real  $t < 0$ . Application of (6.6) gives us

$$\hat{\Omega}_1 \sim -j\omega h \frac{\pi+2}{\pi} \sqrt{\frac{2b}{\pi+4}} (-t)^{-\frac{1}{2}}. \quad (10.4)$$

### 11. Unsteady flow; method of the complex acceleration-potential

The coordinate system and the units of length and time are chosen as in Section 9. We start from the linearized Euler equations

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= -\frac{1}{\varrho} \frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} &= -\frac{1}{\varrho} \frac{\partial p}{\partial y}. \end{aligned} \quad (11.1)$$

In these equations we have linearized with respect to  $q_c=1$  the velocity on the cavity in the steady case. We introduce the function  $\tilde{\varphi}$  by means of

$$\tilde{\varphi} = -\frac{p-p_c}{\varrho}. \quad (11.2)$$

Hence (11.1) can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= \frac{\partial \tilde{\varphi}}{\partial x}, \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} &= \frac{\partial \tilde{\varphi}}{\partial y}. \end{aligned} \quad (11.3)$$

$\tilde{\varphi}$  is called the acceleration-potential. As a result of the incompressibility and irrotationality  $\tilde{\varphi}$  is a harmonic function. We can now introduce its conjugate harmonic function  $\tilde{\psi}$ , satisfying the relations

$$\begin{aligned} \frac{\partial \tilde{\varphi}}{\partial x} &= \frac{\partial \tilde{\psi}}{\partial y}, \\ \frac{\partial \tilde{\varphi}}{\partial y} &= -\frac{\partial \tilde{\psi}}{\partial x}. \end{aligned} \quad (11.4)$$

Now  $F(z) = \tilde{\varphi} + i\tilde{\psi}$  is a holomorphic function of  $z = x + iy$  for which we have

$$\frac{\partial F}{\partial z} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial z}. \quad (11.5)$$

$F(z)$  is the complex acceleration-potential. Now we may formulate the linearized cavity problem of Section 9 as the following mixed boundary-value problem for  $F(z)$ .

Determine  $F(z)$ , holomorphic in the upper half of the  $z$ -plane and assuming on the real axis the following boundary values:

$$\begin{aligned} -\infty < x < 0, & \quad \psi = 0 \quad \text{or} \quad \text{Im } F = 0, \\ 0 < x < l, & \quad \varphi = 0 \quad \text{or} \quad \text{Re } F = 0, \\ l < x < \infty, & \quad \psi = 0 \quad \text{or} \quad \text{Im } F = 0. \end{aligned} \quad (11.6)$$

The general solution with integrable singularities is

$$F(z) = a \sqrt{\frac{z-l}{z}} + b \sqrt{\frac{z}{z-l}}, \quad (11.7)$$

where  $a$  and  $b$  are quantities dependent only on  $t$  and real with respect to  $i$ .

*Remark.* This  $a$  is different from the  $a$  used in Part I. According to the second linearization we can write

$$\begin{aligned} F &= F_0 + F_1 \quad \text{with} \quad F_1 = \hat{F}_1 e^{i\omega t}, \\ a &= a_0 + a_1, & a_1 &= \hat{a}_1 e^{i\omega t}, \\ b &= b_0 + b_1, & b_1 &= \hat{b}_1 e^{i\omega t}. \end{aligned} \quad (11.8)$$

We identify  $F_0$  and  $w_0$ . This is in accordance with the foregoing equations. Thus

$$F_0 = -\frac{\sigma}{4} \left[ \sqrt{\frac{z-l_0}{z}} + \sqrt{\frac{z}{z-l_0}} \right]. \quad (11.9)$$

For  $F_1$  we now get the expression

$$\begin{aligned} F_1 &= -a_0 \frac{\lambda_1}{2} \frac{1}{\sqrt{z(z-l_0)}} + b_0 \frac{\lambda_1}{2} \frac{1}{\sqrt{z(z-l_0)}} \frac{1}{z-l_0} + \\ &+ a_1 \sqrt{\frac{z-l_0}{z}} + b_1 \sqrt{\frac{z}{z-l_0}}, \quad \text{where } \lambda_1 = l - l_0. \end{aligned} \quad (11.10)$$

In this expression for  $F_1$  three unknowns, *viz.*  $\lambda_1$ ,  $a_1$  and  $b_1$ , still occur. For their determination we have the following three conditions:

$$1. \quad F_1 = 0 \quad \text{at} \quad z = \infty, \quad (11.11)$$

*i.e.*, the perturbation pressure vanishes at infinity.

2. The velocity must be a single-valued function. Naturally this requirement is fulfilled for the horizontal component of the perturbation velocity  $u_1$  by virtue of the symmetry of the problem; there is no vortex sheet in the wake. Thus Condition 2 expresses the fact that there is no line of discontinuity for the vertical component of the perturbation velocity  $v_1$ .

3. The unsteady part of the singularity for  $F(z)$  at  $z=0$  and the given unsteady part of the singularity for  $w(z)$  there must satisfy an appropriate relation.

As to Condition 3, we shall prove that the unsteady parts of the singularities for  $F(z)$  and  $w(z)$  at  $z=0$  are the same.

According to (11.5) we have

$$j\omega \hat{w}_1 + \frac{\partial \hat{w}_1}{\partial z} = \frac{\partial \hat{F}_1}{\partial z}, \quad (11.12)$$

from which it follows that

$$\begin{aligned} \hat{w}_1 &= e^{-j\omega z} \int_{-\infty}^z e^{j\omega \zeta} \frac{\partial \hat{F}_1}{\partial \zeta} d\zeta \\ &= \hat{F}_1(z) - j\omega \int_{-\infty}^z e^{j\omega \zeta} F_1(\zeta) d\zeta. \end{aligned} \quad (11.13)$$

Use has been made of (11.11). Since the integral in the last member of (11.13) is finite at  $z=0$ ,  $\hat{w}_1$  and  $\hat{F}_1$  have the same behavior at  $z=0$ . *q.e.d.*

Equating the strengths of the two singularities, we obtain

$$\hat{a}_1 \sqrt{l_0} + \frac{a_0 \hat{\lambda}_1}{2\sqrt{l_0}} = \hat{A}_1 \sqrt{l_0} + A_0 \frac{\hat{l}_1}{2\sqrt{l_0}}. \quad (11.14)$$

This is the mathematical expression of Condition 3. Application of Condition 1 gives us

$$\hat{a}_1 + \hat{b}_1 = 0. \quad (11.15)$$

We shall now formulate Condition 2 mathematically. Since  $\hat{F}_1(z)$  is a single-valued function, we can infer from (11.13) that the single-valuedness of  $\hat{w}_1$  is equivalent to

$$\int_C e^{j\omega \zeta} \hat{F}_1(\zeta) d\zeta = 0, \quad (11.16)$$

where  $C$  is a contour surrounding the cut from 0 to  $l$  in the  $z$ -plane. Since  $a_0 = A_0 = -\frac{1}{4}\sigma$ , we can write (11.10) as

$$\begin{aligned} \hat{F}_1(z) &= \hat{a}_1 \sqrt{\frac{z-l_0}{z}} + \hat{b}_1 \sqrt{\frac{z}{z-l_0}} + \frac{\sigma \hat{\lambda}_1}{8} \frac{1}{\sqrt{z(z-l_0)}} - \\ &- \frac{\sigma \hat{\lambda}_1}{8} \sqrt{\frac{z}{z-l_0}} \frac{1}{z-l_0} = -\frac{\hat{a}_1 l_0}{\sqrt{z(z-l_0)}} + \frac{\sigma \hat{\lambda}_1}{4} \frac{d}{dz} \sqrt{\frac{z}{z-l_0}}, \end{aligned} \quad (11.17)$$

where use has been made of (11.15). Condition (11.16) is now

$$\hat{a}_1 l_0 \int_C \frac{e^{j\omega z}}{\sqrt{z(z-l_0)}} dz + \frac{\sigma \hat{\lambda}_1 j\omega}{4} \int_C e^{j\omega z} \sqrt{\frac{z}{z-l_0}} dz = 0, \quad (11.18)$$

or

$$\hat{a}_1 l_0 \int_0^{l_0} \frac{e^{j\omega x}}{\sqrt{x(l_0-x)}} dx + \frac{\sigma \hat{\lambda}_1 j\omega}{4} \int_0^{l_0} e^{j\omega x} \sqrt{\frac{x}{l_0-x}} dx = 0. \quad (11.19)$$

Replacing  $x$  by  $l_0 - x$  as variable of integration in the integrals, we get

$$\hat{a}_1 l_0 \int_0^{l_0} \frac{e^{-j\omega x}}{\sqrt{x(l_0-x)}} dx + \frac{\sigma \hat{\lambda}_1 j\omega}{4} \int_0^{l_0} e^{-j\omega x} \sqrt{\frac{l_0-x}{x}} dx = 0, \quad (11.20)$$

or, using (6.15) and (6.16),

$$\hat{a}_1 J_0\left(\omega \frac{l_0}{2}\right) + \frac{\sigma \hat{\lambda}_1 j \omega}{8} \left[ J_0\left(\omega \frac{l_0}{2}\right) + j J_1\left(\omega \frac{l_0}{2}\right) \right] = 0. \quad (11.21)$$

From (9.20) and (9.24) we can derive

$$\hat{A}_1 J_0\left(\omega \frac{l_0}{2}\right) + \frac{\sigma \hat{\lambda}_1 j \omega}{8} \left[ J_0\left(\omega \frac{l_0}{2}\right) + j J_1\left(\omega \frac{l_0}{2}\right) \right] = 0. \quad (11.22)$$

Combination of (11.14), (11.21) and (11.22) yields

$$\hat{a}_1 = \hat{A}_1 \quad \text{and} \quad \hat{\lambda}_1 = \hat{l}_1. \quad (11.23)$$

This means that both methods, *viz.*, the method of the complex acceleration-potential and the method of the complex velocity, lead to the same results. This is important, because the method of the complex acceleration-potential is preferable in more complicated problems.

## 12. The drag

We calculate the drag by applying the momentum theorem to the interior of the contour indicated in Figure 5.  $C_\epsilon$  represents the linearized version of the

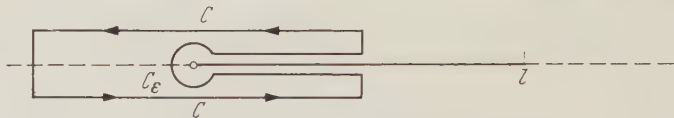


Fig. 5. Contour for the calculation of the drag

flat plate. The whole contour is denoted by  $C$ . We use the method of the complex velocity.

It is clear by virtue of the symmetry with respect to the  $x$ -axis that the component of the force perpendicular to the main stream vanishes. Since the horizontal sections of  $C$  give no contribution to the  $x$ -component of the force, and since the contribution of the vertical end sections can be made arbitrarily small by making the contour thinner, the momentum theorem can be written as

$$\int_{C_\epsilon} (\phi - \phi_c) dy = \varrho \int_C (1 + u) v dx, \quad (12.1)$$

where  $\phi$  is the pressure exerted by the fluid on the flat plate and the direction of  $C_\epsilon$  and  $C$  is chosen positive, *i.e.*, such that the interior of  $C$  lies on the left.

Since the drag  $D = \int_{C_\epsilon} (\phi - \phi_c) dy$ , we have

$$\begin{aligned} D &= \varrho \int_C (1 + u) v dx = -\operatorname{Im} \varrho \int_C w dz - \operatorname{Im} \varrho \int_C \frac{w^2}{2} dz \\ &= -\operatorname{Im} \varrho \int_{C_\epsilon} \frac{w^2}{2} dz = -\pi \varrho \operatorname{Re} \{ \operatorname{res.} w^2 \}_{z=0}. \end{aligned} \quad (12.2)$$

By virtue of (8.15)

$$w_0 \sim -i \sqrt{\frac{2b}{\pi+4}} z^{-\frac{1}{2}} \quad \text{at } z=0,$$

and by virtue of (10.4)

$$\hat{w}_1 \sim -i j \omega h \frac{\pi+2}{\pi} \sqrt{\frac{2b}{\pi+4}} z^{-\frac{1}{2}} \quad \text{at } z=0.$$

Hence we have

$$w = -i \sqrt{\frac{2b}{\pi+4}} \left[ 1 + j\omega h \frac{\pi+2}{\pi} e^{j\omega t} \right] z^{-\frac{1}{2}} \quad \text{at } z=0. \quad (12.3)$$

Substituting in (12.2), we find

$$D = \varrho 2b \frac{\pi}{\pi+4} \left[ 1 + 2j\omega h \frac{\pi+2}{\pi} e^{j\omega t} \right], \quad (12.4)$$

and

$$C_D = \frac{D}{\frac{1}{2}\varrho U^2 2b} = \frac{2\pi}{\pi+4} \left[ 1 + 2j\omega h \frac{\pi+2}{\pi} e^{j\omega t} \right]. \quad (12.5)$$

In (12.5) we have neglected the factor  $(1+\sigma)$  in accordance with the order of approximation we have used throughout. In [13] we have given a formula from which it can be seen that for the steady case the insertion of the factor  $(1+\sigma)$  — already known as the  $(1+\sigma)$  rule — gives the correct next order of approximation.

### 13. Conclusions

In the foregoing sections we have shown that a linearized cavity theory can be formulated in such a way that it becomes a first order approximation to the reentrant-jet theory. For the proof we have chosen the simple example of a flat plate normal to the direction of the main stream since in that case the calculations are as simple as possible. The advantages of this choice are slightly diminished by the fact that we could compare our linearized theory with the first-order approximation to the unsteadily perturbed reentrant-jet flow only for rather large values of  $\omega$ , as discussed in Section 6. We had to make this restriction because the flat plate performs an oscillation in the direction of the main stream. For a lifting plate, oscillating normally to the main flow direction, the restriction concerning  $\omega$  drops out. With respect to the behavior of the flow at the end of the cavity, however, which is the important point in the steady case as well as in the unsteady case, we may safely assume that it is not influenced by the shape of the forebody in the special example chosen.

We have assumed that the unsteady motion is small with respect to the steady part of the disturbance field caused by the body and the cavity. For the non-linear theory, *i.e.*, the reentrant-jet theory, this assumption was necessary in order to attack the problem in a reasonable way, and for the linearized theory it has enabled us to arrive at explicit results. Moreover we may expect that the condition of small unsteady perturbations will be satisfied in many operating conditions. The hydrodynamic stability of the cavity flow was not considered in this paper. It may be determined, however, from the results that we have obtained for the case of harmonic motion.

We have shown that for the linearized cavity flow the method of the complex acceleration-potential leads to the same result as the method of the complex velocity, provided appropriate additional conditions are applied. In the case of the second method these conditions may be derived easily by comparing the linearized flow with the reentrant-jet flow. A condition which seems to be new is the one relating the pressure at infinity and the pressure on the cavity. From our analysis it is clear that in using the method of the complex acceleration-potential it must be required explicitly that the normal component of the per-

turbation velocity should be continuous in the wake. Since this condition has been overlooked hitherto, we mention it here.

The linearization of the vertical flat plate into a singularity, which may be considered as a slight generalization of linearized theory as used thus far enables us to treat more complicated problems in a relatively simple way. *E.g.*, the effects of gravity on a cavity have been dealt with recently in that way.

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# *Bounds on the Dissipation of Energy in Steady Flow of a Viscous Incompressible Fluid around a Body Rotating within a Finite Region*

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## 1. Introduction

The problem of calculating the torque on bodies of revolution in a viscous incompressible fluid in steady flow has received considerable attention. Such calculations are needed in designing and calibrating rotational viscometers. With few exceptions, previous investigations have used STOKES' equations of slow flow, in which the inertial effects of the fluid are ignored. The attempts made to take account of these inertial effects have usually depended on a perturbation technique in which deviations from Stokes flow are expanded in powers of some Reynolds number\*. Such methods appear to be valid for very slow flows, but they suffer from the difficulty that the range of validity is not given by the method and can be determined empirically at best.

When inertial effects can validly be ignored, so that STOKES' linearized theory applies, it is well known that dissipation of energy, and thus also the torque, can be bounded both above and below by certain functionals of a trial function\*\*. What is more, if the true Stokes-flow velocity is selected as the trial function, these bounds converge, and the actual torque is determined.

This paper explores the possibilities of extending these bounding techniques to the non-linear case where inertial terms are considered in full. The Stokes-flow solution is used as a trial function, and upper and lower bounds are found for the torque on a rotating figure of revolution in a finite fixed container in terms of the torque of the Stokes-flow solution, but only for the case that a quantity having the character of a Reynolds number is less than one. One valuable feature of the bounding techniques for Stokes-flow does not carry over in this generalization to the non-linear case; if the actual flow is used for a trial function in calculating the bounds given in this paper, the upper and lower bounds do not converge. The question of existence of a steady flow is not investigated.

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\* See for example DI FRANCIA: Bull. Unione Matematica Italiana, Ser. 3, 5 (1950).

\*\* See Appendix I.

The principal results of this research are twofold:

1. A rigorous maximum error of the Stokes-flow torque is displayed in terms of a uniquely defined quantity with the character of a Reynolds number.
2. A technique is suggested for studying the hydrodynamics of highly non-linear steady flows. When equations are extremely difficult to solve, one is grateful for the information contained in inequalities.

## 2. The Upper Bound on Dissipation

The configuration to be studied consists of a figure of revolution rotating about its axis of symmetry and immersed in an incompressible fluid which is itself confined by a fixed outer wall; see Fig. 1. The region filled with fluid, assumed finite, will be denoted by  $R$ ; the rigid rotating surface will be denoted by  $S_1$ ; while the fixed surface will be denoted  $S_2$ .  $S$  will denote the boundary of  $R$ , the sum of  $S_1$  and  $S_2$ .

The Navier-Stokes equation for an incompressible fluid can be written

$$\varphi_{i,j,j} = 0$$

(1) where

$$\varphi_{ij} = -p \delta_{ij} + 2\mu \bar{d}_{ij}(\mathbf{u}) - \rho u_i u_j.$$

In this and following equations, repeated indices will indicate summation, and the usual tensor notation for Cartesian coordinate systems is used;  $p$  the pressure,  $\mu$  the coefficient of viscosity,  $\rho$  the density of the fluid, and  $\delta_{ij}$  the Kronecker delta. The  $i^{\text{th}}$  Cartesian component of the velocity vector  $\mathbf{u}$  of the fluid is written  $u_i$ , and all subscripts following a comma indicate differentiation with respect to the corresponding Cartesian coordinate. In this notation, the rate of deformation tensor of the vector velocity field  $\mathbf{u}$  is defined by

$$(2) \quad d_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

By GREEN'S transformation, one can obtain

$$(3) \quad \int_S v_i \varphi_{ij} ds_j = \int_R v_{i,j} \varphi_{ij} d\tau$$

where  $\mathbf{v}$  is a vector field, continuous in  $R$  and continuously differentiable throughout the interiors of a finite number of volumes whose sum is  $R$ . The volume element in  $R$  is  $d\tau$ , and  $ds$  is the surface element directed toward the exterior of  $R$ .

Equation (1) implies

$$(4) \quad \int_S v_i \varphi_{ij} ds_j = - \int_S p v_i ds_i + 2\mu \int_S v_i \bar{d}_{ij}(\mathbf{u}) ds_j - \rho \int_S v_i u_i u_j ds_j.$$

But  $\mathbf{u} \cdot d\mathbf{s} = 0$  on the impermeable bounding surfaces. We now specify that  $\mathbf{v} = \mathbf{u}$  on  $S$  so that equation (4) becomes

$$(5) \quad \int_{S_1} v_i \varphi_{ij} ds_j = 2\mu \int_{S_1} u_i \bar{d}_{ij}(\mathbf{u}) ds_j,$$

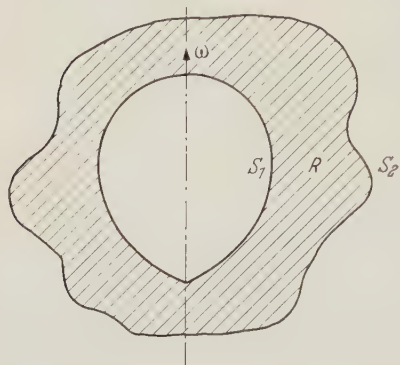


Fig. 1. A cross-sectional diagram of the geometry considered

and this quantity is just the rate of doing work on the fluid in  $R$ . We can call this quantity  $\omega M$  where  $\omega$  is the angular velocity of the surface  $S_1$  and  $M$  is the torque acting on  $S_1$ .

Equations (3) and (5) with the symmetry of  $\boldsymbol{\varphi}$  imply that

$$(6) \quad \omega M = \int_R v_{i,j} \varphi_{ij} d\tau = \int_R d_{ij}(\mathbf{v}) \varphi_{ij} d\tau.$$

By further restricting  $\mathbf{v}$  to be solenoidal, *i.e.*,  $\int v_i ds_i = 0$  over any closed surface in  $R$ , we obtain

$$(7) \quad \omega M = 2\mu \int_R d_{ij}(\mathbf{v}) d_{ij}(\mathbf{u}) d\tau - \rho \int_R u_i u_j d_{ij}(\mathbf{v}) d\tau.$$

Partial integration of the second integral on the right side of equation (7) yields

$$(8) \quad \int_R u_i u_j d_{ij}(\mathbf{v}) d\tau = - \int_R v_i u_j u_{i,j} d\tau.$$

In virtue of the identity

$$(9) \quad \int_R v_i u_j u_{j,i} d\tau = \frac{1}{2} \int_S u_j u_j v_i ds_i = 0,$$

equations (7), (8) and (9) may be combined to give

$$(10) \quad \omega M = 2\mu \int_R d_{ij}(\mathbf{v}) d_{ij}(\mathbf{u}) d\tau + 2\rho \int_R v_i u_j d_{ij}(\mathbf{u}) d\tau.$$

The restrictions placed on the vector field  $\mathbf{v}$  up to this point are the following:

1. Continuous in  $R$  and continuously differentiable in a finite number of volumes whose sum is  $R$ .
2. Converges to  $\mathbf{u}$  on  $S$ .
3. Solenoidal in  $R$ .

It is clear that we may choose  $\mathbf{v}$  to be the solution to the Stokes-flow problem whenever the latter exists, although it is not necessary for  $\mathbf{v}$  to satisfy any differential equation other than that its divergence shall vanish almost everywhere, as is implied by its solenoidal character.

It can be shown that

$$(11) \quad \omega M = 2\mu \int_R d_{ij}(\mathbf{u}) d_{ij}(\mathbf{u}) d\tau,$$

which is just the statement that  $\omega M$  is equal to the dissipation of energy in the fluid. By analogy, we define  $M_0$ , the torque of the trial vector field, by

$$(12) \quad \omega M_0 = 2\mu \int_R d_{ij}(\mathbf{v}) d_{ij}(\mathbf{v}) d\tau.$$

Then, applying SCHWARZ'S inequality to each term on the right of equation (10) with some manipulation results in

$$(13) \quad \omega M \leq \omega \sqrt{M M_0} + \rho \mathcal{V} \sqrt{\frac{2\omega M}{\mu} \int_R u^2 d\tau},$$

where  $\mathcal{V}^2$  is the maximum value of  $v_i v_i$  in  $R^*$ .

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\* When  $\mathbf{v}$  is an axially symmetric Stokes-flow solution, it can be shown that it assumes the magnitude  $\mathcal{V}$  on the boundary of  $R$ .

To find an upper bound for  $M$ , it remains to bound  $\int_R u^2 d\tau$  from above. To do this we consider GREEN's transformation,

$$(14) \quad \int_S u_i \psi_{ij} ds_j = \int_R u_{i,j} \psi_{ij} d\tau + \int_R u_i \psi_{i,j,j} d\tau,$$

where  $\psi$  is any suitably smooth tensor defined in  $R$  and on the boundary  $S$ . In particular, let  $\psi_{ij} = u_i x_j + 2u_j x_i$ ; then

$$(15) \quad \int_S (u_i u_i x_j + 2u_i u_j x_i) ds_j = 4 \int_R u_i x_j d_{ij}(\mathbf{u}) d\tau + 5 \int_R u^2 d\tau.$$

The surface conditions on  $\mathbf{u}$  may be used to reduce this to

$$(16) \quad \int_{S_1} u^2 x_j ds_j = 4 \int_R u_i x_j d_{ij}(\mathbf{u}) d\tau + 5 \int_R u^2 d\tau,$$

which, upon application of SCHWARZ's inequality and some manipulation, yields

$$(17) \quad 5 \int_R u^2 d\tau - 2\mathcal{R} \sqrt{\int_R u^2 d\tau} \sqrt{\frac{2\omega M}{\mu}} \leq I$$

where  $\mathcal{R}$  is the maximum radius in  $R$  from the coordinate center, and we have used the definition

$$(18) \quad I = \int_{S_1} u^2 x_i ds_i.$$

Inequality (17) can be solved to give

$$(19) \quad \sqrt{\int_R u^2 d\tau} \leq \sqrt{\frac{2\omega M}{\mu}} \frac{\mathcal{R}}{5} \left( 1 + \sqrt{1 + \frac{5I\mu}{2\omega M \mathcal{R}^2}} \right).$$

By combining this with inequality (13), one obtains

$$(20) \quad \sqrt{M} \leq \sqrt{M_0} + \frac{2\varrho \mathcal{V}}{5\mu} \mathcal{R} \sqrt{M} \left( 1 + \sqrt{1 + \frac{5I\mu}{2\omega M \mathcal{R}^2}} \right).$$

It is easily seen that if

$$(21) \quad N_R = \frac{4\varrho \mathcal{V} \mathcal{R}}{5\mu} < 1,$$

inequality (20) implies a finite upper bound on  $M$ . This upper bound can be calculated numerically for any particular geometry. With the loss of some sharpness, however, we can solve (20) explicitly. In Appendix II it is demonstrated that  $I$  is negative; therefore inequality (20) can be shortened to give

$$(22) \quad \sqrt{M} \leq \frac{\sqrt{M_0}}{1 - N_R} \quad \text{when } N_R < 1.$$

By observing that  $(1+a)^{\frac{1}{2}} \leq 1 + \frac{a}{2}$ , one can sharpen this result somewhat to give

$$(23) \quad \sqrt{M} \leq \frac{\sqrt{M_0}}{1 - N_R} + \frac{5\mu I N_R}{8\omega \mathcal{R}^2 \sqrt{M_0}}.$$

Undoubtedly even sharper approximations to the solution of inequality (20) can be obtained, but further efforts in this direction seem hardly worthwhile in light of the fact that it can be solved numerically for any particular geometry. Inequalities (22) and (23) are rigorous upper bounds on  $\sqrt{M}$ .

### 3. The Lower Bound on Dissipation

A lower bound may be obtained more briefly. We choose a trial tensor  $\psi$ , symmetric and such that

$$(24) \quad \psi_{ij,j} = q_{,i} \quad \text{in } R,$$

where  $q$  is a bounded scalar field in  $R$ . Tensors satisfying this requirement can be simply constructed in the following way\*. Choose a symmetric tensor field, say  $\mathbf{a}$ , with finite second derivatives defined in  $R$ . Then

$$(25) \quad \psi_{ij} = \varepsilon_{ikl} \varepsilon_{jmn} a_{ln,km} + q \delta_{ij}$$

is a solution of equation (24), where  $\varepsilon_{ijk}$  is the usual permutation tensor density.

By GREEN'S transformation, one can write

$$(26) \quad \int_S u_i \psi_{ij} ds_j = \int_R u_{i,j} \psi_{ij} d\tau + \int_R u_i q_{,i} d\tau.$$

Because of the symmetry of  $\psi$  and the boundary condition on  $\mathbf{u}$ , equation (26) can be written

$$(27) \quad \int_{S_1} u_i \psi_{ij} ds_j = \int_R d_{ij}(\mathbf{u}) \psi_{ij} d\tau.$$

Further, by SCHWARZ'S inequality, we have

$$(28) \quad \omega M = 2\mu \int_R d_{ij}(\mathbf{u}) d_{ij}(\mathbf{u}) d\tau \geq 2\mu \frac{\left( \int_{S_1} u_i \psi_{ij} ds_j \right)^2}{\int_R \psi_{ij} \psi_{ij} d\tau}.$$

Inequality (28) constitutes a calculable lower bound on the energy dissipation in  $R$ . In particular, we may choose for the trial tensor  $\psi$  the rate of deformation tensor of the Stokes-flow velocity field satisfying the prescribed boundary conditions, viz,

$$(29) \quad \psi_{ij} = d_{ij}(\mathbf{v}), \quad q = \frac{p'}{2\mu}$$

where  $p'$  is the pressure field associated with the Stokes-flow solution. Then

$$(30) \quad \omega M \geq 2\mu \frac{\left( \int_{S_1} v_i d_{ij}(\mathbf{v}) ds_j \right)^2}{\int_R d_{ij}(\mathbf{v}) d_{ij}(\mathbf{v}) d\tau}$$

and hence

$$(31) \quad M \geq M_0.$$

### 4. Conclusions

1. It is interesting to observe that the greatest lower bound contained in inequality (30) is inequality (31). This fact is clearly implied by inequality (41) of Appendix I. In general, this analysis is too rough to expect convergence of the upper and lower bounds for any trial functions, a fact which is in contrast

\* See DORN & SCHILD: Quart. Appl. Math. **14**, 209–213 (1956) and TRUESDELL: Arch. rational Mech. Anal. **4**, 1–29 (1959).

to the Stokes-flow case discussed in Appendix I. An interesting question is whether a modification of this bounding technique could be sufficient to assure convergence of the bounds.

2. The upper bound on the dissipation is lost if  $N_R$  is greater than one. In a later paper, it will be shown that this fact is a result of the roughness of the approximations employed in the analysis and has no hydrodynamic significance.

3. Inequalities (13) and (17) are enough to specify an upper bound on total kinetic energy of the flow. A simple and less sharp form of this bound is

$$(32) \quad \varrho \int_R u^2 d\tau \leq \frac{8\varrho R^2 \omega}{25\mu} \frac{M_0}{(1-N_R)^2},$$

which is valid when  $N_R$  is less than one.

4. The quantity  $N_R$  is a dimensionless number which may be considered a Reynolds number, uniquely defined and calculable for all cylindrically symmetric cases and all cases where the Stokes-flow solution is known. It serves as a rigorous criterion for the error introduced in calculating dissipation by ignoring the inertial terms in the Navier-Stokes equation. Put another way, it specifies conditions which guarantee that the dissipation calculated from a Stokes-flow solution will be within a specified range of the dissipation according to the full Navier-Stokes steady flow solution.

5. Although this analysis has been done for the case where the inner surface rotates and the outer surface is fixed, it is clear that a simple extension of the analysis can take into account rotation of the outer surface as well.

## Appendix I

The Stokes-flow equation for an incompressible viscous fluid can be written

$$(33) \quad \varphi_{i,j,j} = 0, \quad \text{where} \quad \varphi_{ij} = -p \delta_{ij} + 2\mu d_{ij}(\mathbf{u}).$$

If  $\mathbf{u}$  is the solution to this equation and  $\mathbf{v}$  is a smooth divergenceless vector field\* equal to  $\mathbf{u}$  on  $S$ , then by GREEN's transformation

$$(34) \quad \int_{S_1} v_i \varphi_{ij} ds_j = \int_R v_{i,j} \varphi_{ij} d\tau,$$

where  $\mathbf{v}$  and  $\mathbf{u}$  satisfy the boundary conditions of the previous problem. By equation (33) this becomes

$$(35) \quad \omega M_0 = 2\mu \int_R d_{ij}(\mathbf{u}) d_{ij}(\mathbf{v}) d\tau.$$

An application of SCHWARZ's inequality to equation (35) yields

$$(36) \quad (\omega M_0)^2 \leq 4\mu^2 \int_R d_{ij}(\mathbf{u}) d_{ij}(\mathbf{u}) d\tau \int_R d_{ij}(\mathbf{v}) d_{ij}(\mathbf{v}) d\tau.$$

---

\* More precisely,  $\mathbf{v}$  must satisfy the three conditions written out previously following equation (10).

Finally, in light of equation (5), we have

$$(37) \quad \omega M_0 \leq 2\mu \int_R \dot{d}_{ij}(\mathbf{v}) \dot{d}_{ij}(\mathbf{v}) d\tau,$$

which constitutes an upper bound\* on  $\omega M_0$ .

For a lower bound on the dissipation of energy, we select as a trial function a smooth symmetric tensor field,  $\psi$ , which is a solution of

$$(38) \quad \psi_{ij,i} = q_{,i},$$

where  $q$  is a continuously differentiable scalar field in  $R$ , converging to a finite value on  $S$ . Then by GREEN's transformation

$$(39) \quad \int_S u_i \psi_{ij} ds_j = \int_R u_{i,j} \psi_{ij} d\tau + \int_R u_i q_{,i} d\tau.$$

But in view of the boundary condition on  $\mathbf{u}$  and the symmetry of  $\psi$ ,

$$(40) \quad \int_{S_1} u_i \psi_{ij} ds_j = \frac{1}{2} \int_R \dot{d}_{ij}(\mathbf{u}) \psi_{ij} d\tau.$$

Then by SCHWARZ's inequality and equation (5) we have

$$(41) \quad \omega M_0 \geq 2\mu \frac{\left( \int_{S_1} u_i \psi_{ij} ds_j \right)^2}{\int_R \psi_{ij} \psi_{ij} d\tau}.$$

This inequality should be compared with inequality (28).

Inequalities (37) and (41) constitute respectively upper and lower bounds of the dissipation of energy associated with a Stokes-flow solution of the problem. In particular, if the trial vector  $\mathbf{v}$  and the trial tensor  $\psi$  are taken as the Stokes-flow velocity and rate of deformation tensor, respectively, these bounds will converge to the dissipation associated with  $\mathbf{u}$ .

Although these bounding principles for the linear Stokes equation seem to be well known, I have been unable to find in the literature a good reference for the upper bound. Almost identical, however, are the principles of DIRICHLET and THOMSON which bound the Dirichlet integral of solutions to the Laplace equation\*\*.

## Appendix II

The integral  $I$  has been defined in equation (18), viz:

$$(18) \quad I = \int_{S_1} u^2 x_i ds_i.$$

\* This bound can be recognized as a form of the well known Helmholtz Theorem; see for example SERRIN: Handbuch der Physik VIII/1, p. 258. Berlin-Göttingen-Heidelberg: Springer 1959.

\*\* See for example J. B. DIAZ: Upper and Lower Bounds for Quadratic Integrals, and at a Point, for Solutions of Linear Boundary Value Problems. Proceedings of a Symposium on Boundary Problems in Differential Equations, Madison Wisconsin, 1959.

The velocity on the surface  $S_1$  is known, since  $S_1$  is the surface of a rigid rotating body with angular velocity specified. If  $\omega$  is the angular velocity vector of  $S_1$ , we have

$$(42) \quad u_i = \varepsilon_{ijk} \omega_j x_k \quad \text{on } S_1.$$

Let us consider equation (42) as extending the vector field  $\mathbf{u}$  into the interior of the volume bounded by  $S_1$ . Then by GREEN's transformation, noting that  $\omega$  is a fixed vector, we get

$$(43) \quad I = -5 \int_{R'} \varepsilon_{ijk} \varepsilon_{ilm} \omega_j \omega_l x_k x_m d\tau,$$

where  $R'$  is the region bounded by  $S_1$  and the negative sign is due to the fact that the outward drawn surface element of the region  $R$  is the negative of the outward drawn surface element of the region  $R'$ . We observe, finally, that

$$(44) \quad I = -5 \int_{R'} u^2 d\tau \leq 0,$$

which proves  $I$  to be negative.

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## CONTENTS

KELLER, J. B., The Shape of the Strongest Column . . . . .	275
PAYNE, L. E., & H. F. WEINBERGER, An Optimal Poincaré Inequality for Convex Domains . . . . .	286
NITSCHKE, J., & J. C. C. NITSCHKE, Error Estimates for the Numerical Solution of Elliptic Differential Equations . . . . .	293
NITSCHKE, J. C. C., & J. NITSCHKE, Fehlerabschätzung für die numerische Berechnung von Integralen, die Lösungen elliptischer Differential- gleichungen enthalten . . . . .	307
GEURST, J. A., Some Investigations of a Linearized Theory for Un- steady Cavity Flows . . . . .	315
KEARSLEY, E. A., Bounds on the Dissipation of Energy in Steady Flow of a Viscous Incompressible Fluid around a Body Rotating within a Finite Region . . . . .	347